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Symmetry Preservation by a Compatible Staggered Lagrangian Scheme Using the Control-Volume Discretization Method in r–z Coordinate

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Abstract. This paper aims at developing a control volume staggered Lagrangian scheme in r–z coordinate that preserves symmetry property. To achieve this goal, the support operator method is first utilized to derive the compatible discretization that satisfies the Geometrical Conservation Law (GCL) and momentum and total energy conservation property. We further introduce a method of source term treatment to recover the spherical symmetry of the current scheme. It is shown that the developed scheme has the benefit of maintaining the momentum and total energy conservation. The equi-angular grid, randomly distorted polar grid, and Cartesian grid are considered for one-dimensional spherical flow simulations. Also, an extension to the non-spherical flow is presented. The results confirm the good performance of the developed scheme.

AMS subject classifications: 65M08, 76M12

Key words: Symmetry preservation, compatibility, control-volume scheme, staggered Lagrangian scheme.

1 Introduction

Lagrangian methods [1–6] and the associated Arbitrary Lagrangian Eulerian (ALE) methods [7–11] nowadays constitute a standard approach to deal with high-speed compressible multimaterial flow problems. There are two main kinds of Lagrangian methods, namely the staggered and cell-centered Lagrangian methods. The staggered methods solve the governing equations in a non-conservative form. Numerous attempts have been conducted on them to make advances on the real-life applications [12–15]. The cell-centered methods, on the other hand, have gained much attention in recent years. They

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are solved in a conservative form so that the solutions tend to a weak solution of the continuous problem.

The importance of symmetry property in Lagrangian methods is well recognized, especially when the flows are described in cylindrical geometry. For example, in a one-dimensional spherical flow, loss of symmetry will cause both the grid and variable fields to depart from a spherical shape as they evolve. Besides, in the case of extensions to non-symmetric flow problems, uncertainties can arise as to whether a nonsymmetric result comes from the physical process itself or from numerical error. Currently, cell-centered methods make better progress in symmetry preserving than staggered methods in that the conservative properties are preserved simultaneously [16–18]. However, any cell-centered method has a flaw that severe inaccuracy might occur in strong expansions, which is a potential obstacle for real-life applications [19].

Realizing the deficiency of cell-centered methods, it is important to promote the research of staggered methods as well. The focus of this work is symmetry preservation in a pure staggered Lagrangian framework. A simple way to achieve this goal is the employment of a "Cartesian form" of the momentum equation in cylindrical geometry. Methods developed in this way are usually called area-weighted methods [8, 20] because the integrations involved are performed in an area rather than a true volume. Area-weighted schemes are widely applied to real problem simulations, especially when discretization is done in a compatible manner. However, they can suffer from limitations due to their lack of momentum conservation. Another natural way to perform discretization is through the control-volume method. Caramana and Whalen [21] introduced a control-volume staggered method that preserves spherical symmetry and is not limited to an equi-angular polar grid. In their method, the gradient operator is modified in such a way that it fully recovers spherical symmetry and introduces only a very small change in the simulation results. Nevertheless, they found that this method, similar to the area-weighted method, does not strictly conserve momentum. In more recent work by Váchal and Wendroff [22], a so-called staggered GCS scheme was established using the control-volume method to preserve symmetry and total energy and reduce the GCL error to the order of the entropy error. Not enough is the investigation of total momentum conservation. Margolin and Shashkov [23] adopted a novel strategy to implement discretization on a curvilinear grid, and the scheme that they developed is able to preserve symmetry even on a nonuniform polar grid. Dobrev et al. [24] also developed a scheme on a curvilinear grid. Their method exactly conserves total energy and excels at preserving symmetry, while also avoiding the generation of spurious symmetry breaking near the rotation axis even for nonuniform grids.

Based on literature researches, the situation of symmetry preservation is less satisfactory in the non-curvilinear staggered methods, taking into account the conservative properties. To remedy the deficiency, the compatible discretization, whose weak consistency has been proved strictly [25], is first utilized in the current study. we use for reference to the methods [16–18] in which the symmetry and conservative properties are satisfied simultaneously by invoking a method of source term treatment. Consequently,