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A Remark on the J. L. Lions Lemma and its Applications in a Variable Exponent Sobolev Space

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Abstract. In the author's previous paper, we considered the equivalent conditions with $W^{-m,p(\cdot)}$-version ($m\geq 0$ integer) of the J. L. Lions Lemma, where $p(\cdot)$ is a variable exponent. In this paper, we directly derive $W^{-m,p(\cdot)}$-version of the J. L. Lions Lemma. Therefore, we can use all of the equivalent conditions. As an application, we derive the generalized Korn inequality. Furthermore, we consider the relation to other fundamental results.

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Key words: J. L. Lions Lemma, de Rham Theorem, Korn inequality, variable exponent Sobolev spaces.

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1 Introduction

Assume that Ω is a bounded domain of $\mathbb{R}^d$ with a Lipschitz-continuous boundary $\partial\Omega$, Ω is locally on the same side of Γ and $p\in\mathcal{P}_+^{\log}(\Omega)$ is a variable exponent. In the previous paper Aramaki [3], we derived the following theorem on the equivalent conditions with the J. L. Lions Lemma.

Theorem 1.1. *Let Ω be a bounded domain of $\mathbb{R}^d$ with a Lipschitz-continuous boundary $\Gamma=\partial\Omega$ and Ω be locally on the same side of $\partial\Omega$, and let $m\geq 0$ be a integer and $p\in\mathcal{P}_+^{\log}(\Omega)$. Then the following (a), (b), ..., and (f) are equivalent.*

(a) *Classical J. L. Lions Lemma: if $f\in W^{-m-1,p(\cdot)}(\Omega)$ satisfies $\boldsymbol{\nabla} f\in \boldsymbol{W}^{-m-1,p(\cdot)}(\Omega)$, then $f\in W^{-m,p(\cdot)}(\Omega)$.*

(b) *The Nečas inequality: there exists a constant $C_0=C_0(m,p(\cdot),\Omega)$ such that*

$$\|f\|_{W^{-m,p(\cdot)}(\Omega)}\leq C_0(\|f\|_{W^{-m-1,p(\cdot)}(\Omega)}+\|\boldsymbol{\nabla} f\|_{\boldsymbol{W}^{-m-1,p(\cdot)}(\Omega)})\quad \textit{for all } f\in W^{-m,p(\cdot)}(\Omega).$$

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(c) *The operator grad has a closed range:* $\mathbf{grad}(W^{-m,p(\cdot)}(\Omega)/\mathbb{R})$ *is a closed subspace of* $\boldsymbol{W}^{-m-1,p(\cdot)}(\Omega)$.

(d) *A coarse version of the de Rham Theorem: for any* $\boldsymbol{h}\in\boldsymbol{W}^{-m-1,p(\cdot)}(\Omega)$, *there exists a unique* $[\pi]\in W^{-m,p(\cdot)}(\Omega)/\mathbb{R}$, *where* $[\pi]$ *denotes the class in* $W^{-m,p(\cdot)}(\Omega)/\mathbb{R}$ *with the representative* π, *such that* $\boldsymbol{h}=\nabla\pi$ *in* $\boldsymbol{W}^{-m-1,p(\cdot)}(\Omega)$ *if and only if*

$$\langle \boldsymbol{h},\boldsymbol{v}\rangle_{\boldsymbol{W}^{-m-1,p(\cdot)}(\Omega),\boldsymbol{W}_0^{m+1,p'(\cdot)}(\Omega)}=0 \quad \textit{for all } \boldsymbol{v}\in\boldsymbol{W}_0^{m+1,p'(\cdot)}(\Omega,\mathrm{div}0).$$

(e) *The operator* div *is surjective: the operator*

$$\mathrm{div}:\boldsymbol{W}_0^{m+1,p'(\cdot)}(\Omega)\to\dot{W}_0^{m,p'(\cdot)}(\Omega).$$

is continuous and surjective. In addition, if $f\in\dot{\mathcal{D}}(\Omega)$, *then there exists* $\boldsymbol{u}_f\in\boldsymbol{\mathcal{D}}(\Omega)$ *such that* $\mathrm{div}\boldsymbol{u}_f=f$.

Consequently, for any $f\in\dot{W}_0^{m,p'(\cdot)}(\Omega)$, *there exists a unique*

$$[\boldsymbol{u}_f]\in\boldsymbol{W}_0^{m+1,p'(\cdot)}(\Omega)/\mathbf{Ker}\mathrm{div},$$

where $\mathbf{Ker}\mathrm{div}=\boldsymbol{W}_0^{m+1,p'(\cdot)}(\Omega,\mathrm{div}0)$ *and* $[\boldsymbol{u}_f]$ *denotes the class in* $\boldsymbol{W}_0^{m+1,p'(\cdot)}(\Omega)/\mathbf{Ker}\mathrm{div}$ *with the representative* $\boldsymbol{u}_f$ *such that* $\mathrm{div}[\boldsymbol{u}_f]=f$ *in* Ω. *Therefore, the operator*

$$\mathrm{div}:\boldsymbol{W}_0^{m+1,p'(\cdot)}(\Omega)/\mathbf{Ker}\mathrm{div}\to\dot{W}_0^{m,p'(\cdot)}(\Omega)$$

is linear, continuous and bijective. Hence, by the Banach open mapping theorem, there exists a constant $C_1=C_1(m,p(\cdot),\Omega)>0$ *such that*

$$\|[\boldsymbol{u}_f]\|_{\boldsymbol{W}_0^{m+1,p'(\cdot)}(\Omega)/\mathbf{Ker}\mathrm{div}}\le C_1\|f\|_{W^{m,p'(\cdot)}(\Omega)} \quad \textit{for all} f\in\dot{W}_0^{m,p'(\cdot)}(\Omega).$$

(f) *The J. L. Lions Lemma: if* $f\in\mathcal{D}'(\Omega)$ *satisfies* $\boldsymbol{\nabla} f\in\boldsymbol{W}^{-m-1,p(\cdot)}(\Omega)$, *then we can find that* $f\in W^{-m,p(\cdot)}(\Omega)$.

When $p(\cdot)=\text{const.}=2$ and $m=0$, Amrouche *et al.* [1] derived this theorem in L^2-framework. Various results are drawn from the classical J. L. Lions Lemma, see, for example, Boyer and Fabrie [5] and Ciarlet [6]. Aramaki [4] derived an improvement to the case where $p(\cdot)=\text{const.}=p$ $(1<p<\infty)$ and $m\ge 0$ is an integer. We can prove that the classical Nečas inequality (b) holds directly (cf. Amrouche and Girault [2, Theorem 2.3]). Consequently if Ω is a bounded domain with a Lipschitz-continuous boundary, $m\ge 0$ is an integer and $p(\cdot)=p=\text{const.}$, then all of (a)-(f) are usable. We applied the result to the existence of a weak solution to the Maxwell-Stokes type problem in the previous paper [4].

For general $p\in\mathcal{P}_+^{\log}(\Omega)$ and $m=0$, the Nečas inequality (b) of Theorem 1.1 holds (Diening *et al.* [7, Theorem 14.3.18]), thus all of (a)-(f) are usable in the case $m=0$. However,