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Neighbor Sum Distinguishing Total Chromatic Number of Graphs with Lower Average Degree

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Abstract. For a given simple graph $G=(V(G),E(G))$, a proper total-k-coloring $c: V(G)\cup E(G)\to\{1,2,\ldots,k\}$ is neighbor sum distinguishing if $f(u)\neq f(v)$ for each edge $uv\in E(G)$, where $f(v)=\sum_{wv\in E(G)}c(wv)+c(v)$. The smallest integer k in such a coloring of G is the neighbor sum distinguishing total chromatic number, denoted by $\chi''_{\Sigma}(G)$. It has been conjectured that $\chi''_{\Sigma}(G)\leq\Delta(G)+3$ for any simple graph G. Let $mad(G)=\max\{\frac{2|E(H)|}{|V(H)|}:H\subseteq G\}$ be the maximum average degree of G. In this paper, by using the famous Combinatorial Nullstellensatz, we prove $\chi''_{\Sigma}(G)\leq\max\{9,\Delta(G)+2\}$ for any graph G with $mad(G)<4$. Furthermore, we characterize the neighbor sum distinguishing total chromatic number for every graph G with $mad(G)<4$ and $\Delta(G)\geq 8$.

AMS subject classifications: 53A07, 53C24, 53C40

Key words: Neighbor sum distinguishing total coloring, combinatorial nullstellensatz, maximum average degree

1 Introduction

All graphs mentioned in this paper are undirected, finite and simple. For a graph G, we denote its vertex set, edge set, maximum degree and minimum degree by $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$, respectively. For $v\in V(G)$, the degree of v, denoted by $d(v)$, is the number of edges incident with v. A vertex v is called a k-vertex, a k^+-vertex and a k^--vertex if $d(v)=k$, $d(v)\geq k$ and $d(v)\leq k$, respectively. A k-neighbor of v is a neighbor of v with degree k. Let $d_k(v)$, $d_{k^+}(v)$ and $d_{k^-}(v)$ be the number of neighbors of v with degree k, at least k and at most k in G, respectively. For $v\in V(G)$ and $U\subseteq V(G)$, the number of neighbors of v in U is denoted by $d_U(v)$. A 1-vertex is also said to be a *leaf*. The average degree $ad(G)$ of a graph G is defined as $\frac{2|E(G)|}{|V(G)|}$. The maximum average degree $mad(G)$

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of a graph G is the maximum of the average degrees of its subgraphs. For notation and terminology not defined in this paper, see [2].

A proper total-k-coloring of a graph $G=(V(G),E(G))$ is an assignment $c:V(G)\cup E(G)\to\{1,\ldots,k\}$ such that $c(x)\neq c(y)$ for each pair of adjacent or incident elements $x,y\in V(G)\cup E(G)$. For $v\in V(G)$, let $f(v)=\sum_{wv\in E(G)}c(wv)+c(v)$ be the sum of the colors assigned to the edges incident with v and the color of v. If $f(u)\neq f(v)$ for each edge $uv\in E(G)$, then c is called as a neighbor sum distinguishing total-k-coloring or a tnsd-k-coloring of G for short. The smallest integer k in such a coloring of G is the neighbor sum distinguishing total chromatic number of G, denoted by $\chi''_{\Sigma}(G)$.

Given a total-k-coloring c of G, let $S(v)$ denote the set of colors of edges incident with v and the color of v. If $S(u)\neq S(v)$ for each edge $uv\in E(G)$, then c is called as an adjacent vertex distinguishing total-k-coloring. The smallest integer k in such a coloring of G is the adjacent vertex distinguishing total chromatic number of G, denoted by $\chi''_a(G)$. In 2005, Zhang *et al.* [14] put forward the following conjecture.

Conjecture 1.1 ([14]). *For any graph G with at least two vertices, $\chi''_a(G)\leq\Delta(G)+3$.*

Zhang *et al.* [14] proved the conjecture for graphs which are paths, cycles, fans, wheels, stars, complete graphs, bipartite complete graphs and trees. Chen [3] and Wang [11] proved independently the Conjecture 1.1 for graphs with $\Delta(G)\leq 3$. In 2008, Wang and Wang [13] showed that if G is a graph with $mad(G)<3$, then $\chi''_a(G)\leq\max\{7,\Delta(G)+2\}$. In 2012, Huang and Wang [7] proved that $\chi''_a(G)\leq\max\{14,\Delta(G)+3\}$ for planar graphs. In 2014, Wang and Huang [12] proved that $\chi''_a(G)\leq\max\{15,\Delta(G)+2\}$ for planar graphs. In 2020, Chang *et al.* [4] proved that $\chi''_a(G)\leq\max\{11,\Delta(G)+3\}$ for planar graphs.

In 2011, Pilśniak and Woźniak [9] introduced the concept of tnsd-k-coloring and put forward the following conjecture.

Conjecture 1.2 ([9]). *For any graph G with at least two vertices, $\chi''_{\Sigma}(G)\leq\Delta(G)+3$.*

Conjecture 1.2 implies Conjecture 1.1. Pilśniak and Woźniak [9] confirmed the Conjecture 2 for cycles, complete graph, subcubic graphs and bipartite graphs. In 2013, Li *et al.* [8] proved that conjecture holds for K_4-minor free graphs. In 2014, Dong *et al.* [5] showed that if G is a graph with $mad(G)<3$, then $\chi''_{\Sigma}(G)\leq\max\{7,\Delta(G)+2\}$. In 2017, Qiu *et al.* [10] proved that if G is a graph with $mad(G)<\frac{9}{2}$, then $\chi''_{\Sigma}(G)\leq\max\{11,\Delta(G)+3\}$. In 2020, Hocquard and Przybylo [6] proved that if G is a graph with $mad(G)<\frac{14}{3}$, then $\chi''_{\Sigma}(G)\leq\max\{11,\Delta(G)+3\}$.

In this paper, we prove the following result:

Theorem 1.1. *Let G be a graph with $mad(G)<4$.*
(1) *Then $\chi''_{\Sigma}(G)\leq\max\{9,\Delta(G)+2\}$.*
(2) *If $\Delta(G)\geq 8$, then $\chi''_{\Sigma}(G)=\Delta(G)+1$ if and only if G without adjacent $\Delta(G)$-vertices.*

The girth $g(G)$ of a graph G is the smallest length of the cycles in G. We get the following corollary from Theorem 1.1 for planar graph.