Analysis in Theory and Applications
Anal. Theory Appl., Vol. **39**, No. 2 (2023), pp. 120-146

DOI: 10.4208/ata.OA-2022-0009
June 2023

The Solutions with Prescribed Asymptotic Behavior for the Exterior Dirichlet Problem of Hessian Equations

Limei Dai*

School of Mathematics and Information Science, Weifang University, Weifang, Shandong 261061, China

Received 7 February 2022; Accepted (in revised version) 12 April 2022

Abstract. In this paper, we consider the exterior Dirichlet problem of Hessian equations

$$\sigma_k(\lambda(D^2u)) = g(x)$$

with g being a perturbation of a general positive function at infinity. The existence of the viscosity solutions with generalized asymptotic behavior at infinity is established by the Perron's method which extends the previous results for Hessian equations. By the solutions of Bernoulli ordinary differential equations, the viscosity subsolutions and supersolutions are constructed.

Key Words: Hessian equations, exterior Dirichlet problem, asymptotic behavior.

AMS Subject Classifications: 35J60, 35J25, 35B40, 35D40

1 Introduction

In this paper, we will study the Dirichlet problem of Hessian equations in exterior domains

$$\sigma_k(\lambda(D^2u)) = g(x), \qquad x \in \mathbb{R}^n \backslash \overline{\Omega}, \tag{1.1a}$$

$$u = \phi(x), \qquad x \in \partial\Omega, \tag{1.1b}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded set, $n \geq 3$, $\lambda(D^2u)$ denotes the eigenvalues $\lambda_1, \cdots, \lambda_n$ of the Hessian matrix D^2u,

$$\sigma_k(\lambda(D^2u)) = \sum_{1\leq i_1<\cdots<i_k\leq n} \lambda_{i_1}\cdots\lambda_{i_k}$$

is the k-th elementary symmetric function, $k = 1, \ldots, n$, $g \in C^0(\mathbb{R}^n)$ is positive and $\phi \in C^2(\partial\Omega)$. Note that for $k = 1$, (1.1a) is the Poisson equation $\Delta u = g(x)$, which is a linear

*Corresponding author. *Email address:* lmdai@wfu.edu.cn (L. Dai)

elliptic equation; for $k=n$, (1.1a) is the notable Monge-Ampère equation $\det D^2u=g(x)$ which is a fully nonlinear elliptic equation.

In 2003, Caffarelli and Li [8] ($n\geq 3$) investigated the exterior Dirichlet problem of Monge-Ampère equation

$$\begin{cases}\det D^2u=1, & x\in\mathbb{R}^n\backslash\overline{\Omega},\\ u=\phi, & x\in\partial\Omega,\end{cases}\tag{1.2}$$

and obtained that if Ω is a smooth, bounded, strictly convex open subset and $\phi\in C^2(\partial\Omega)$, then for any given $b\in\mathbb{R}^n$ and any given $n\times n$ real symmetric positive definite matrix A with $\det A=1$, there exists some constant $\hat{c}$ depending only on n,Ω,ϕ,b and A, such that for every $c>\hat{c}$ there exists a unique function $u\in C^\infty(\mathbb{R}^n\backslash\overline{\Omega})\cap C^0(\overline{\mathbb{R}^n\backslash\Omega})$, which satisfies (1.2) and

$$\limsup_{|x|\to\infty}\left(|x|^{n-2}\left|u(x)-\left(\frac{1}{2}x^TAx+b\cdot x+c\right)\right|\right)<\infty.$$

Since then, there have been extensive studies of the exterior problem for the fully nonlinear elliptic equations. In 2011, the first author and Bao [15] ($n\geq 3$), the first author [13] ($n\geq 3$) studied the exterior Dirichlet problem of Hessian equation

$$\sigma_k(\lambda(D^2u))=1\tag{1.3}$$

and got the existence and uniqueness of the viscosity solutions with the asymptotic behavior

$$\limsup_{x\to\infty}\left(|x|^{\Upsilon-2}\left|u(x)-\left(\frac{c_*}{2}|x|^2+c\right)\right|\right)<\infty,\tag{1.4}$$

where $\Upsilon=n$ or k, $c\in\mathbb{R}$ and

$$c_*=(1/C_n^k)^{\frac{1}{k}}.$$

Then in 2012, Wang and Bao [30] studied the necessary and sufficient conditions on existence of radially symmetric solutions for the Dirichlet problem outside a unit ball $B_1=B_1(0)$,

$$\begin{cases}\sigma_k(\lambda(D^2u))=1, & x\in\mathbb{R}^n\setminus\overline{B_1},\\ u=\text{constant}, & x\in\partial B_1,\end{cases}$$

with the asymptotic behavior

$$\begin{aligned} u(x)&=\frac{c_*}{2}|x|^2+c+\mathcal{O}(|x|^{2-n}), && |x|\to\infty, && n\geq 3,\\ u(x)&=\frac{1}{2}|x|^2+\frac{d}{2}\ln|x|+c+\mathcal{O}(|x|^{2-n}), && |x|\to\infty, && n=2,\end{aligned}$$

where $c,d\in\mathbb{R}$. Compared with the asymptotic behavior of solutions of Monge-Ampère equations, it seems more suitable that the solutions of Hessian equations should tend to