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# A Scale-Invariant Fifth Order WCNS Scheme for Hyperbolic Conservation Laws

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**Abstract.** In this article, a robust, effective, and scale-invariant weighted compact nonlinear scheme (WCNS) is proposed by introducing descaling techniques to the nonlinear weights of the WCNS-Z/D schemes. The new scheme achieves an essentially non-oscillatory approximation of a discontinuous function (ENO-property), a scale-invariant property with an arbitrary scale of a function (Si-property), and an optimal order of accuracy with smooth function regardless of the critical point (Cp-property). The classical WCNS-Z/D schemes do not satisfy Si-property intrinsically, which is caused by a loss of sub-stencils' adaptivity in the nonlinear interpolation of a discontinuous function when scaled by a small scale factor. A new nonlinear weight is devised by using an average of the function values and the descaling function, providing the new WCNS schemes (WCNS-Zm/Dm) with many attractive properties. The ENO-property, Si-property and Cp-property of the new WCNS schemes are validated numerically. Results show that the WCNS-Zm/Dm schemes satisfy the ENO-property and Si-property, while only the WCNS-Dm scheme satisfies the Cp-property. In addition, the Gaussian wave problem is solved by using successively refined grids to verify that the optimal order of accuracy of the new schemes can be achieved. Several one-dimensional shock tube problems, and two-dimensional double Mach reflection (DMR) problem and the Riemann IVP problem are simulated to illustrate the ENO-property and Si-property of the scale-invariant WCNS-Zm/Dm schemes.

**AMS subject classifications**: 65N06, 35Q35

**Key words**: WCNS, descaling function, scale-invariant, ENO-property, Cp-property.

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# 1 Introduction

In the past few decades, computational fluid dynamics (CFD) has received considerable attention. Numerical schemes enormously influence the computational accuracy, robustness, and efficiency. Despite the extra complexity of high-order numerical schemes [1], they can capture discontinuities with high resolution, and portray complex flow details more realistically and accurately compared with second-order schemes [2–5]. However, the discontinuity in the high-speed flow requires that numerical schemes need to be properly dissipative to enable robust computations and suppress spurious oscillations on the one hand. The numerical oscillations will contaminate the flow field and result in the computational divergence. On the other hand, the high-resolution require that dissipation and dispersion errors of numerical schemes need to be minimized. These two aspects put forward almost contradictory requirements for numerical schemes.

Linear schemes are usually used in computations involving smooth flows, such as dispersion-relation-preserving schemes [6]. However, linear schemes are not suitable to the numerical simulation of discontinuities (such as shock waves). There are two methods used for the capturing the discontinuity. One is the shock-fitting method, and the other is the shock-capturing method. The shock-fitting method is difficult to be extended to flow field calculations of complex flow structures [7, 8]. Therefore, shock-capturing methods are more widely used and developed. Among shock-capturing methods, a popular family of schemes is the total variance diminishing (TVD) scheme proposed by Harten [9]. Subsequently, Van Leer constructed the monotonic upstream-centered scheme for conservation laws (MUSCL) based on the TVD scheme, which can reduce numerical dissipation and dispersion errors and achieve a higher resolution [10, 11]. However, the TVD scheme suffers from a loss of accuracy at the critical point when calculating complex flows, like turbulence. To improve the accuracy and ensure no degradation, higher-order schemes are developed to solve the difficulties encountered in second-order schemes. High-order shock-capturing schemes are more efficient than second-order schemes in attaining the same accuracy [12]. They can reduce the dissipation and dispersion errors and achieve a higher resolution to some extent [13], but this object is very challenging since the satisfaction of high numerical resolution and strong computational robustness are contradictory [5, 14]. In addition, higher-order schemes provide good parallel scalability and efficiency compared against low-order schemes for the same grids [15–17].

The Weighted Essentially Non-Oscillatory (WENO) scheme is one of the most successful high-order shock-capturing schemes. This scheme was originally proposed by Liu et al. [18] and is an improved version of the ENO scheme put forward by Harten et al. [19, 20]. It has been widely used in various flows [21, 22]. Furthermore, Jiang and Shu [23] constructed third and fifth order WENO schemes and designed nonlinear weights and smoothness indicators. Following the idea of the WENO, two essential variants of WENO-type schemes have been developed. One is WENO-M. Henrick et al. [24] extended the WENO scheme and proposed the new mapped WENO (WENO-