Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **15**, No. 5, pp. 1233-1255

DOI: 10.4208/aamm.OA-2022-0188
October 2023

Arbitrarily High-Order Energy-Preserving Schemes for the Camassa-Holm Equation Based on the Quadratic Auxiliary Variable Approach

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Received 10 July 2022; Accepted (in revised version) 11 September 2022

Abstract. In this paper, we present a quadratic auxiliary variable (QAV) technique to develop a novel class of arbitrarily high-order energy-preserving algorithms for the Camassa-Holm equation. The QAV approach is first utilized to transform the original equation into a reformulated QAV system with a consistent initial condition. Then the reformulated QAV system is discretized by applying the Fourier pseudo-spectral method in space and the symplectic Runge-Kutta methods in time, which arrives at a class of fully discrete schemes. Under the consistent initial condition, they can be rewritten as a new fully discrete system by eliminating the introduced auxiliary variable, which is rigorously proved to be energy-preserving and symmetric. Ample numerical experiments are conducted to confirm the expected order of accuracy, conservative property and efficiency of the proposed methods. The presented numerical strategy makes it possible to directly apply a special class of Runge-Kutta methods to develop energy-preserving algorithms for a general conservative system with any polynomial energy.

AMS subject classifications: 65M06, 65M70

Key words: Camassa-Holm equation, quadratic auxiliary variable, high-order energy-preserving schemes, symplectic Runge-Kutta methods.

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1 Introduction

In this paper, we consider the Camassa-Holm (CH) equation

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \quad (x,t) \in [a,b] \times (0,T], \tag{1.1}$$

with periodic boundary condition

$$u(x,t) = u(b-a+x,t), \quad t \in [0,T], \tag{1.2}$$

and initial condition

$$u(x,0) = u_0(x), \quad x \in [a,b], \tag{1.3}$$

where u is the fluid velocity in the x direction (or equivalently the height of the fluid's free surface above a flat bottom). The CH equation is a model for the unidirectional propagation of shallow water waves [8] as well as a model for nonlinear waves in cylindrical hyperelastic rods [17]. It has a bi-Hamiltonian structure with an infinite number of conserved functionals [20, 21]. In particular, the system (1.1)-(1.3) possesses the following three conserved quantities

$$\mathcal{I} = \int_a^b u dx, \quad \mathcal{M} = \int_a^b (u^2 + u_x^2) dx, \quad \mathcal{H} = \frac{1}{2} \int_a^b (u^3 + u u_x^2) dx. \tag{1.4}$$

where $\mathcal{I}$, $\mathcal{M}$ and $\mathcal{H}$ correspond to mass, momentum and energy of the original problem, respectively.

Because of the rich mathematical structure and interesting properties of the CH equation [8, 14, 20, 21, 38], it is very important to develop geometric numerical integrators or structure-preserving algorithms for solving the CH equation accurately. In the early days, some spatial structure-preserving algorithms were proposed for the CH equation, including finite difference method [10, 30], Fourier spectral or pseudo-spectral method [36, 37] and local discontinuous Galerkin method [48], etc. Some adaptive spatial approximations were presented to capture the peakon efficiently [2, 19, 48]. There also have been developed more fully discrete structure-preserving algorithms, such as symplectic or multi-symplectic integrators [6,12,45,51], momentum-preserving methods [13,40,42] and energy-preserving algorithms [13, 23, 41]. Most existing conservative schemes are based on discrete variational derivative methods [16, 22], which are often fully implicit. Recently, Hong et al. developed two linear-implicit momentum-conserving schemes, which could preserve the original momentum exactly [31]. According to the polarised discrete gradient methods and the Kahan's method, Eidnes et al. constructed two linearly implicit energy-preserving schemes for the CH equation [18]. Jiang et al. applied the energy quadratization approach [44, 49] to obtain two linear energy-preserving schemes [33, 34]. At each time step, the linearly implicit schemes only require to solve a linear system, which leads to considerably lower costs than the implicit ones. However, these linear-implicit energy-preserving methods only maintain a modified energy, which is not equal