Journal of Computational Mathematics
Vol.41, No.5, 2023, 797–816.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2111-m2020-0177

A LINEARLY-IMPLICIT ENERGY-PRESERVING ALGORITHM FOR THE TWO-DIMENSIONAL SPACE-FRACTIONAL NONLINEAR SCHRÖDINGER EQUATION BASED ON THE SAV APPROACH*

Yayun Fu
School of Science, Xuchang University, Xuchang 461000, China
Email: fyyly@xcu.edu.cn
Wenjun Cai
Jiangsu Key Laboratory for NSLSCS,
Jiangsu Collaborative Innovation Center of Biomedial Functional Materials,
School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China
Email: caiwenjun@njnu.edu.cn
Yushun Wang[1)]
Jiangsu Key Laboratory for NSLSCS,
Jiangsu Collaborative Innovation Center of Biomedial Functional Materials,
School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China
Email: wangyushun@njnu.edu.cn

Abstract

The main objective of this paper is to present an efficient structure-preserving scheme, which is based on the idea of the scalar auxiliary variable approach, for solving the two-dimensional space-fractional nonlinear Schrödinger equation. First, we reformulate the equation as an canonical Hamiltonian system, and obtain a new equivalent system via introducing a scalar variable. Then, we construct a semi-discrete energy-preserving scheme by using the Fourier pseudo-spectral method to discretize the equivalent system in space direction. After that, applying the Crank-Nicolson method on the temporal direction gives a linearly-implicit scheme in the fully-discrete version. As expected, the proposed scheme can preserve the energy exactly and more efficient in the sense that only decoupled equations with constant coefficients need to be solved at each time step. Finally, numerical experiments are provided to demonstrate the efficiency and conservation of the scheme.

Mathematics subject classification: 35R11, 65M70.
Key words: Fractional nonlinear Schrödinger equation, Hamiltonian system, Scalar auxiliary variable approach, Structure-preserving algorithm.

1. Introduction

Due to the memory and genetic property of fractional operators, fractional differential equations are more suitable to model various scientific and engineering problems with long-range temporal cumulative memory effects and spatial interactions than integer order, thus they are widely implemented in the fields of biomedical engineering [1, 2], physics [3], and hydrological applications [4]. The space fractional nonlinear Schrödinger (NLS) equation, introduced in

* Received July 1, 2020 / Revised version received February 9, 2021 / Accepted November 9, 2021 / Published online August 30, 2022 /
1) Corresponding author

Refs. [5,6] via extending the Feynman path integral to the Lévy integral, is fractional version of classical NLS equation and consider long-range interactions. In quantum mechanics, the fractional NLS equation is more accurate than the integer order in describing how the quantum state of a physical system changes in time [8,15]. In addition, the equation also arises in the continuum limit of a family of discrete models for charge transport in biopolymers like the DNA [35]. For physical realizations or applications of the equation, the readers can refer to Refs. [6,36,37] and references therein for more details. Nowadays, studies for the equation have been considered and many significant achievements have been made [7,8,13].

In this paper, we numerically consider the fractional NLS equation as follows

$$\mathrm{i}\frac{\partial u(\boldsymbol{x},t)}{\partial t} - (-\Delta)^{\frac{\alpha}{2}} u(\boldsymbol{x},t) + \big(V(\boldsymbol{x}) + \beta|u(\boldsymbol{x},t)|^2\big)u(\boldsymbol{x},t) = 0, \qquad t \in (0,T], \tag{1.1}$$

with the initial condition

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega, \tag{1.2}$$

where $1 < \alpha \leq 2$, $\mathrm{i}^2 = -1$, $u(\boldsymbol{x},t)$ is a complex-valued wave function with periodic boundary conditions, $\boldsymbol{x} \subset \mathbb{R}^d$ (d=1, 2) and $u_0(\boldsymbol{x})$ is a given smooth function, $V(\boldsymbol{x})$ is an arbitrary potential function, β is a real constant with positive value for focusing (or attractive) nonlinearity. The fractional Laplacian $-(-\Delta)^{\frac{\alpha}{2}}$ is defined as a pseudo-differential operator with the symbol $|\boldsymbol{\xi}|^\alpha$ in the Fourier space [9]

$$\widehat{(-\Delta)^{\frac{\alpha}{2}}u}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^\alpha \widehat{u}(\boldsymbol{\xi}), \tag{1.3}$$

where $\widehat{u}(\boldsymbol{\xi}) = \int_\Omega u(\boldsymbol{x})e^{-\mathrm{i}\boldsymbol{\xi}\boldsymbol{x}}d\boldsymbol{x}$ denotes the Fourier transform of $u(\boldsymbol{x})$. If $\alpha = 2$, the fractional NLS equation reduces to the standard NLS equation.

As we all known, many classical nonlinear partial differential equations are known to posses some physical quantities that naturally arise from the physical context, such as symplectic and multi-symplectic conservation laws, energy and mass conservation laws and so on. In Ref. [7], Guo derived that system (1.1) with the periodic boundary condition possesses the following fractional energy and mass conservation laws

$$E(t) = E(0), \qquad M(t) = M(0), \tag{1.4}$$

where the energy is defined as

$$E(t) := \int_\Omega \Big[\frac{1}{2}|(-\Delta)^{\frac{\alpha}{4}} u(\boldsymbol{x},t)|^2 - \frac{1}{2}V(\boldsymbol{x})|u(\boldsymbol{x},t)|^2 - \frac{\beta}{4}|u(\boldsymbol{x},t)|^4\Big]d\boldsymbol{x}, \tag{1.5}$$

and the mass has the form

$$M(t) := \int_\Omega |u(\boldsymbol{x},t)|^2 d\boldsymbol{x}. \tag{1.6}$$

The prior researches generally confirmed that the structure-preserving methods conserving one or more intrinsic properties of a given dynamical system, are more superior than the traditional methods in long time stability for numerical simulations [19–22]. Nowadays, investigating the structure-preserving numerical schemes for fractional equations [28–30], especially for the fractional NLS equation [10–12,14,16–18,23], has captured researchers' increasing attention. It is well known that the energy conservation law plays an important role in conservative partial