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A Fast Two-Level Strang Splitting Method for Multi-Dimensional Spatial Fractional Allen-Cahn Equations with Discrete Maximum Principle

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Abstract. Numerical solutions of the multi-dimensional spatial fractional Allen-Cahn equations are studied. After the semi-discretization of the spatial fractional Riesz derivative, a system of nonlinear ordinary differential equations with the Toeplitz structure is obtained. In order to reduce the computational complexity, a two-level Strang splitting method is proposed, where the Toeplitz matrix in the system is represented as the sum of circulant and skew-circulant matrices. Therefore, the method can be quickly implemented by the fast Fourier transform, avoiding expensive Toeplitz matrix exponential calculations. It is shown that the discrete maximum principle of the method is unconditionally preserved. Moreover, the analysis of errors in the infinite norm with second-order accuracy is carried out in both time and space. Numerical tests support the theoretical findings and show the efficiency of the method.

AMS subject classifications: 65F10, 65N22

Key words: Two-level Strang splitting method, circulant and skew-circulant matrix splitting, discrete maximum principle, fast Fourier transform.

1. Introduction

The complexity of anomalous diffusion phenomena in the real world and the rare availability of analytic solutions to fractional diffusion equations led to the development of efficient numerical methods for such equations. In particular, for the equations with the Riesz fractional derivative in space, various discretization schemes have been studied in [1, 31, 34, 41, 43] and the corresponding fast computational algorithms are considered in [18–21, 26, 30, 36, 38].

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In this work, we develop a numerical method for the following high-dimensional spatial fractional Allen-Cahn (SFAC) equations:

$$\begin{aligned} &u_t = \varepsilon^2 \mathscr{L}^{\alpha}_{x^d} u + u - u^3, && x \in \Omega, \quad t \in (0,T], \\ &u(x,0) = u^0(x), && x \in \overline{\Omega}, \\ &u|_{\partial\Omega} = 0, && t \in (0,T], \end{aligned} \tag{1.1}$$

where $\Omega = [a,b]^d$, $d = 2,3$ is two- or three-dimensional domain, $\varepsilon > 0$ an interfacial parameter, and $\mathscr{L}^{\alpha}_{x^d}$ the d-dimensional Riesz fractional operator — cf. [17]. The corresponding one-dimensional operator of order $\alpha_1 \in (1,2)$ is defined by

$$\mathscr{L}^{\alpha}_{x^1} u = \mathscr{L}^{\alpha_1}_{x^{(1)}} u := \frac{1}{-2\cos(\alpha_1 \pi/2)} \left({}_a\mathscr{D}^{\alpha_1}_{x^{(1)}} u + {}_{x^{(1)}}\mathscr{D}^{\alpha_1}_{b} u \right),$$

where the left- and right-side Riemann-Liouville fractional derivatives have the following form:

$$\begin{aligned} {}_a\mathscr{D}^{\alpha}_{x} u &= \frac{1}{\Gamma(2-\alpha)} \cdot \frac{\mathrm{d}^2}{\mathrm{d}x^2} \int_a^x \frac{u(\xi)}{(x-\xi)^{\alpha-1}} \mathrm{d}\xi, \\ {}_x\mathscr{D}^{\alpha}_{b} u &= \frac{1}{\Gamma(2-\alpha)} \cdot \frac{\mathrm{d}^2}{\mathrm{d}x^2} \int_x^b \frac{u(\xi)}{(\xi-x)^{\alpha-1}} \mathrm{d}\xi, \end{aligned}$$

cf. [39]. Consequently, two- and three-dimensional Riesz fractional derivatives are respectively defined by $\mathscr{L}^{\alpha}_{x^2} u = \mathscr{L}^{\alpha_1}_{x^{(1)}} u + \mathscr{L}^{\alpha_2}_{x^{(2)}} u$ and $\mathscr{L}^{\alpha}_{x^3} u = \mathscr{L}^{\alpha_1}_{x^{(1)}} u + \mathscr{L}^{\alpha_2}_{x^{(2)}} u + \mathscr{L}^{\alpha_3}_{x^{(3)}} u$, where $\alpha_1, \alpha_2, \alpha_3 \in (1,2)$ are the orders of the corresponding fractional derivatives.

The Allen-Cahn equations are widely used in many areas, including two-phase incompressible fluids, mean curvature flows, dendritic growth dynamics, image inpainting, and segmentation [8,10,11,23,27,29]. Meanwhile, the time-discrete format of the SFAC equations have been studied in [6,44]. Hou *et al.* [17] proposed a Crank-Nicolson scheme, which is a second-order time discretization method, and demonstrated a discrete maximum principle for the first time. However, this scheme is computationally expensive since an iteration algorithm is used in order to compute nonlinear terms. On the other hand, it preserves the discrete maximum principle and the energy decay conditionally. An operator splitting with alternating direction implicit (splitting-ADI) method was proposed by He *et al.* [16]. This splitting-ADI method is much faster than the previous scheme. It extends to the fourth-order accuracy in time, but it abides the discrete maximum principle by restricting the time step. Liao *et al.* [28] developed a second-order backward differentiation formula with nonuniform grids in time-fractional Allen-Cahn equations and proved this scheme is energy stable and preserves the discrete maximum principle under the time step restriction. Du *et al.* presented a second-order exponential time differencing with the Runge-Kutta method (ETDRK2) [7]. The ETDRK2 preserves the maximum principle unconditionally. However, the method is only applicable to the problems with periodic boundary conditions and the accuracy is low. Chen and Sun [4,5] presented efficient dimensional splitting ETDRK2 and Strang splitting methods for problems with Dirichlet boundary conditions. The