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Linearly Compact Difference Scheme for the Two-Dimensional Kuramoto-Tsuzuki Equation with the Neumann Boundary Condition

Qifeng Zhang[1] and Lu Zhang[2,*]

[1] *Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou, Zhejiang 310018, China*

[2] *School of Mathematics and Statistics, Xuzhou University of Technology, Xuzhou, Jiangsu 221018, China*

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This work is dedicated to Prof. Hai-Wei Sun on the occasion of his 60th birthday

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Abstract. In this paper, we analyze and test a high-order compact difference scheme numerically for solving a two-dimensional nonlinear Kuramoto-Tsuzuki equation under the Neumann boundary condition. A three-level average technique is utilized, thereby leading to a linearized difference scheme. The main work lies in the pointwise error estimate in H^2-norm. The optimal fourth-order convergence order is proved in combination of induction, the energy method and the embedded inequality. Moreover, we establish the stability of the difference scheme with respect to the initial value under very mild condition, however, does not require any step ratio restriction. Extensive numerical examples with/without exact solutions under diverse cases are implemented to validate the theoretical results.

AMS subject classifications: 65M06, 65M12, 65M15

Key words: Kuramoto-Tsuzuki equation, compact difference scheme, pointwise error estimate, stability, numerical simulation.

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*Corresponding author.
Emails: `zhangqifeng0504@zstu.edu.cn` (Q. Zhang), `yulu7517@126.com` (L. Zhang)

1 Introduction

In this paper, we will study a high-order compact difference scheme for the initial-boundary value problem of the two-dimensional nonlinear Kuramoto-Tsuzuki (KT) equation in the form of

$$\begin{cases} u_t=(1+\mathrm{i}c_1)\Delta u+\gamma u-(1+\mathrm{i}c_2)|u|^2u, & (x,y)\in\Omega, \quad 0<t\leq T, & (1.1\text{a}) \\ u(x,y,0)=\varphi(x,y), & (x,y)\in\bar{\Omega}, \quad 0<t\leq T, & (1.1\text{b}) \\ \dfrac{\partial u}{\partial \boldsymbol{\nu}}=0, & 0<t\leq T, & (1.1\text{c}) \end{cases}$$

where u is an unknown complex function. $\mathrm{i}=\sqrt{-1}$, c_1 and c_2 are two real constants, which could characterize linear and nonlinear dispersion respectively, see e.g., [2]. γ is a general parameter, which controls the degree of aggregation of solutions. The calculated domain is on $\Omega=(0,L_1)\times(0,L_2)$, and $\boldsymbol{\nu}$ is the unit normal vector of the boundary Ω. $\partial\Omega$ is the boundary of the domain. $\varphi(x,y)$ is a given function.

The KT equation [6,7] describes the behavior of two branches near the bifurcation point. Many efforts have been made to develop highly effective algorithms for the KT equation in one dimension. For example, Tsertsvadze [18] applied Crank-Nicolson method to establish a nonlinear difference scheme for solving the one-dimensional KT equation with the convergence order $\mathcal{O}(h^{\frac{3}{2}})$ in the sense of discrete L^2-norm. Ivanauskas [5] investigated an effective implicit Crank-Nicolson type weighted scheme for the KT equation and the convergence was proved. Sun [17] constructed a linearized three-level difference scheme, which can be solved by the double-sweep method and proved that it is uniquely solvable and convergent. Sun [13,14,16] further developed several new second-order difference schemes and made detailed analysis at length. Štikonas [12] discussed the root condition of a finite difference scheme for the KT equation. Omrani [11] analyzed the convergence of Galerkin method for the KT equation. Wang et al. [19, 20] respectively used semi-explicit difference scheme and nonlinear difference scheme for solving the KT equation. Dong [2] gave a fourth-order split-step pseudospectral scheme and Hu et al. [4] first proposed several fourth-order compact difference schemes for solving the KT equation.

As far as we know, no research work has been done about the numerical solutions of the high-dimensional KT equation under the Neumann boundary condition. Therefore, it is necessary to develop effective numerical algorithms for the KT equation in high dimension. The studies that have been done for high-dimensional KT equation so far include the following two work. One of them dues to Li et al. [9], who discussed a type of the high-dimensional KT equation with Dirichlet boundary condition by Galerkin finite element method and the optimal error estimates are