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Convergence and Stability of an Explicit Method for Autonomous Time-Changed Stochastic Differential Equations with Super-Linear Coefficients

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Abstract. In this paper, numerical methods for the time-changed stochastic differential equations of the form

$$dY(t)=a(Y(t))dt+b(Y(t))dE(t)+\sigma(Y(t))dB(E(t))$$

are investigated, where all the coefficients $a(\cdot)$, $b(\cdot)$ and $\sigma(\cdot)$ are allowed to contain some super-linearly growing terms. An explicit method is proposed by using the idea of truncating terms that grow too fast. Strong convergence in the finite time of the proposed method is proved and the convergence rate is obtained. The proposed method is also proved to be able to reproduce the asymptotic stability of the underlying equation in the almost sure sense. Simulations are provided to demonstrate the theoretical results.

AMS subject classifications: 60H10, 65C30, 60J60

Key words: Time-changed stochastic differential equations, explicit method, super-linear coefficients, strong convergence, asymptotic stability.

1 Introduction

Time-changed stochastic processes have been attracting increasing attentions in recent years, since they are powerful tools to describe sub-diffusion phenomena and they have the strong connection with fractional partial differential equations [3,25,28,36].

One of the blooming topic in this area is the study on time-changed stochastic differential equations (SDEs). Compared with classical SDEs, time-changed SDEs could be

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used to describe the relatively slower diffusion of particles that may be stuck for a random period of time. In addition, time-changed SDEs have many different probabilistic properties from the classical SDEs [20]. So the analyses of both the underlying solutions and numerical solutions to time-changed SDEs need some new ideas and techniques. Many existing works have been devoted to the investigations on different properties of the underlying solutions to time-changed SDEs. Kobayashi studied the existence and uniqueness of the solution to a large class of time-changed SDEs and proposed the corresponding Itô formula in [20]. Wu, in [37] and [38], investigated different types of stability of time-changed SDEs driven by the time-changed Brownian motion when the coefficients of the underlying equations satisfy the global Lipschitz condition. Liu studied the polynomial stability in the moment sense of time-changed SDEs driven by the time-changed Brownian when some super-linear terms appear in the coefficients in [23]. Zhu et al. discussed the almost sure stability of a class of time-changed SDEs in [42]. Nane and Ni developed the Itô formula for time-changed Lévy processes and studied the stability of time-changed SDEs driven by the time-changed Lévy process in [30] and [31]. Zhang and Yuan proved the Razumikhin-type theorem for hybrid time-changed stochastic function differential equations in [40]. We refer the readers to the monograph [36] by Umarov et al. for the detailed and systematic introduction to time-changed SDEs and their related topics.

Compared with the fruitful results on the underlying equations, the amount of works on numerical methods for time-changed SDEs is relatively less. However, it should be noted that the numerical methods are important for the application of time-changed SDEs, since explicit forms of solutions to most of time-changed SDEs are rarely obtained.

To our best knowledge, the first work that directly discretizes the underlying equation to obtain the numerical approximation is [17], where the Euler-Maruyama (EM) method was proposed for time-changed SDEs of the form

$$dY(t)=b(E(t),Y(t))dE(t)+\sigma(E(t),Y(t))dB(E(t)). \tag{1.1}$$

Under the global Lipschitz condition for the state variables in the coefficients $a(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$, the convergences in both the strong and weak senses were proved in that paper. Afterwards, the semi-implicit EM method [7] and the truncated EM method [24] method were proposed for (1.1), when the global Lipschitz condition is no longer imposed. All the three papers [7,17,24] employed the duality principle that was discovered by Kobayashi in [20]. Briefly speaking, to construct numerical methods for (1.1), one could first construct numerical methods for the classical SDEs driven by the classical Brownian motion of the form

$$dy(t)=b(t,y(t))dt+\sigma(t,y(t))dB(t).$$

Then, due to $Y(t)=y(E(t))$, together with some proper discretization of $E(t)$ one could obtain numerical methods for (1.1).