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Semi-Discrete and Fully Discrete Weak Galerkin Finite Element Methods for a Quasistatic Maxwell Viscoelastic Model

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Abstract. This paper considers weak Galerkin finite element approximations on polygonal/polyhedral meshes for a quasistatic Maxwell viscoelastic model. The spatial discretization uses piecewise polynomials of degree k $(k \geq 1)$ for the stress approximation, degree $k+1$ for the velocity approximation, and degree k for the numerical trace of velocity on the inter-element boundaries. The temporal discretization in the fully discrete method adopts a backward Euler difference scheme. We show the existence and uniqueness of the semi-discrete and fully discrete solutions, and derive optimal a priori error estimates. Numerical examples are provided to support the theoretical analysis.

AMS subject classifications: 35Q74, 65M12, 65M60

Key words: Quasistatic Maxwell viscoelastic model, weak Galerkin method, semi-discrete scheme, fully discrete scheme, error estimate.

1. Introduction

Let $\Omega \subset \mathbb{R}^d (d=2,3)$ be a convex polyhedral domain with boundary $\partial\Omega$, and T be a positive constant. We consider the following quasistatic Maxwell viscoelastic model:

$$-\mathbf{div}\boldsymbol{\sigma} = \boldsymbol{f}, \qquad (x,t) \in \Omega \times [0,T], \tag{1.1a}$$
$$\boldsymbol{\sigma} + \boldsymbol{\sigma}_t = \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u_t}), \qquad (x,t) \in \Omega \times [0,T], \tag{1.1b}$$
$$\boldsymbol{u} = 0, \qquad (x,t) \in \partial\Omega \times [0,T], \tag{1.1c}$$
$$\boldsymbol{u}(x,0) = \phi_0(x), \qquad x \in \Omega, \tag{1.1d}$$
$$\boldsymbol{\sigma}(x,0) = \psi_0(x), \qquad x \in \Omega. \tag{1.1e}$$

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Here $\boldsymbol{u} \in \mathbb{R}^d$ is the displacement field, $\boldsymbol{\sigma} = (\sigma_{ij})_{d\times d}$ the symmetric stress tensor, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{(\boldsymbol{\nabla u}+(\boldsymbol{\nabla u})^{\boldsymbol{T}})}{2}$ the strain tensor, f the body force, $\phi_0(x)$ and $\psi_0(x)$ are initial data, $g_t := \frac{\partial g}{\partial t}$ for any function $g(\boldsymbol{x},t)$, and $\mathbb{C}$ denotes an elastic module tensor such that for any symmetric tensor $\boldsymbol{\tau} = (\tau_{ij})_{d\times d}$ a.e. $x \in \Omega$ one has

$$0 < M_0\boldsymbol{\tau} : \boldsymbol{\tau} \le \mathbb{C}^{-1}\boldsymbol{\tau} : \boldsymbol{\tau} \le M_1\boldsymbol{\tau} : \boldsymbol{\tau}, \tag{1.2}$$

where M_0 and M_1 are two positive constants, and

$$\boldsymbol{\iota} : \boldsymbol{\tau} := \sum_{i=1}^{d}\sum_{j=1}^{d} \iota_{ij}\tau_{ij} \quad \text{for} \quad \boldsymbol{\iota}, \boldsymbol{\tau} \in \mathbb{R}^{d\times d}.$$

Note that for an isotropic elastic medium we have

$$\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u_t}) = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u_t}) + \lambda(\nabla\cdot\boldsymbol{u_t})I,$$

where μ and λ are Lamé constants, and I the identity matrix.

In material science and continuum mechanics, viscoelasticity is the property of materials that exhibit both viscous and elastic characteristic when undergoing deformation. The Maxwell model, characterized by the governing constitutive relation (1.1b), is one of classical models of viscoelasticity (see, e.g. [2, 12–14, 16, 18, 19, 33, 40, 41] for some related works on the development and applications of viscoelasticity theory). These models, including the Kelvin-Voigt model and the Zener model, are represented by different combinations of purely elastic springs, which obey Hooke's law, and purely viscous dashpots, which obey Newton law. The Maxwell model consists of a spring and a dashpot connected in series. We note that the general constitutive law of viscoelasticity can be described in a unified framework by using convolution integrals in time with some kernels [12, 16, 40].

In [5, 6] Carcione *et al.* gave the first numerical simulation of wave propagation in viscoelastic materials, and introduced memory variables to avoid the computation of convolution integrals in the constitutive relation. Janovsky *et al.* [25] applied continuous/discontinuous Galerkin finite element methods to discretize a linear viscoelasticity model involving the hereditary constitutive relations for compressible solids. Ha *et al.* [20] proposed a nonconforming finite element method for a viscoelastic complex model in the space frequency domain. Bécache *et al.* [1] presented a family of mass lumped mixed finite element methods, together with a leap-frog scheme in the time discretization, for the Zener model. In [36–38] Rivière *et al.* analyzed discontinuous Galerkin finite element discretizations of the quasistatic linear viscoelasticity and linear/nonlinear diffusion viscoelastic models, where a Crank-Nicolson temporal scheme is used in the full discretization. Rognes and Winther [39] considered mixed finite element approximations with weak symmetric stresses for the quasistatic Maxwell and Kelvin-Voigt models, where the temporal discretization uses a second backward difference scheme. In [42] Shi and Zhang applied the standard p-order rectangular finite