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Some Exact Results on 4-Cycles: Stability and Supersaturation

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Abstract. Extremal problems on the 4-cycle C_4 played a heuristic important role in the development of extremal graph theory. A fundamental theorem of Füredi states that the Turán number $\mathrm{ex}(q^2+q+1,C_4) \leq (1/2)q(q+1)^2$ holds for every $q \geq 14$, which matches with the classic construction of Erdős-Rényi-Sós and Brown from finite geometry for prime powers q. Very recently, we obtained the first stability result on Füredi's theorem, by showing that for large even q, every (q^2+q+1)-vertex C_4-free graph with more than $(1/2)q(q+1)^2-0.2q$ edges must be a spanning subgraph of a unique polarity graph [20]. Using new technical ideas in graph theory and finite geometry, we strengthen this by showing that the same conclusion remains true if the number of edges is lowered to $(1/2)q(q+1)^2-(1/2)q+o(q)$. Among other applications, this gives an immediate improvement on the upper bound of $\mathrm{ex}(n,C_4)$ for infinite many integers n. A longstanding conjecture of Erdős and Simonovits states that every n-vertex graph with $\mathrm{ex}(n,C_4)+1$ edges contains at least $(1+o(1))\sqrt{n}$ 4-cycles. We proved an exact result and confirmed Erdős-Simonovits conjecture for infinitely many integers n [20]. As the second main result of this paper, we further characterize all extremal graphs for which achieve the ℓ-th least number of copies of C_4 for any fixed positive integer ℓ. This can be extended to more general settings and provides enhancements on the understanding of the supersaturation problem of C_4.

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1 Introduction

Given a graph F, we say a graph is F-free if it does not contain F as a subgraph. The Turán

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number $\mathrm{ex}(n,F)$ of F is the maximum number of edges in an n-vertex F-free graph. Turán type and related extremal problems are the central subjects of extremal graph theory. In this paper, we focus on extremal problems on one of the basic and perhaps most influential objects in this area – the cycle C_4 of length four. (For indistinct notations appeared below, we shall refer readers to Section 2.)

Proposed by Erdős [8] more than 80 years ago, the study of $\mathrm{ex}(n,C_4)$ has a rich history. Reiman showed a general upper bound that [24]

$$\mathrm{ex}(n,C_4)\le\frac{n}{4}\left(1+\sqrt{4n-3}\right).$$

However, it is known that the equality never holds by the Friendship Theorem of Erdős, Rényi and Sós [12]. One can also deduce from the proof of Reiman that if the number of edges in an n-vertex C_4-free graph is close to $(1/2)n^{3/2}$, then almost all vertices have roughly $\sqrt{n}$ neighbors and almost all pairs of vertices have one common neighbor. This suggests that perhaps in principle, the neighborhoods of vertices can be regarded as lines of certain projective plane (see Section 2.2). Indeed, using orthogonal polarity graphs (see Section 2.3) constructed from finite projective planes, Erdős *et al.* [12] and Brown [4] proved a lower bound that

$$\mathrm{ex}\left(q^2+q+1,C_4\right)\ge\frac{1}{2}q(q+1)^2 \tag{1.1}$$

for all prime powers q. These two results together imply an asymptotic formula that

$$\mathrm{ex}(n,C_4)=\left(\frac{1}{2}+o(1)\right)n^{\frac{3}{2}}.$$

Determining the exact value of $\mathrm{ex}(n,C_4)$ in general seems to be extremely difficult and far beyond reach. On the other hand, Erdős conjectured (e.g. [9]) that the orthogonal polarity graph is optimal; that is, the inequality in (1.1) should be replaced by an equality for all prime powers q. Füredi [15] first confirmed this for $q=2^k$ in 1983, by showing

$$\mathrm{ex}\left(q^2+q+1,C_4\right)\le\frac{1}{2}q(q+1)^2$$

holds for all even q. In 1996, Füredi [17] proved that the same upper bound holds for all $q\ge 14$. We summarize his results as following.

Theorem 1.1 (Füredi [15,17])**.** *If $q\notin\{1,7,9,11,13\}$, then*

$$\mathrm{ex}\left(q^2+q+1,C_4\right)\le\frac{1}{2}q(q+1)^2.$$

Hence, for all prime powers $q\ge 14$,

$$\mathrm{ex}\left(q^2+q+1,C_4\right)=\frac{1}{2}q(q+1)^2.$$