East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.240222.210722

Vol. **13**, No. 1, pp. 119-139
February 2023

An Efficient EXCMG-Newton Method Combined with Fourth-Order Compact Schemes for Semilinear Poisson Equations

Pinxia Wu[1], Kejia Pan[1,*], Weiwei Ling[1] and Dongdong He[2]

[1]*School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha, Hunan 410083, China.*
[2]*School of Science and Engineering, The Chinese University of Hong Kong, Shenzhen, Guangdong 518172, China.*

Received 24 February 2022; Accepted (in revised version) 21 July 2022.

Abstract. A fast solver for nonlinear systems arising from fourth-order compact finite difference schemes for two-dimensional semilinear Poisson equations is constructed. Applying the extrapolation and bi-quartic interpolation to two numerical solutions from the previous two levels of grids, we determine a suitable initial guess for the Newton iterations on the next finer grid. It is fifth-order accurate, which substantially reduces the number of Newton iterations required. Moreover, an extrapolated solution of sixth-order accuracy can be easily constructed on the whole fine grid. Numerical results suggest that the method is much more efficient than the existing multigrid methods for semilinear problems.

AMS subject classifications: 65N06, 65N55

Key words: Semilinear Poisson equation, fourth-order compact scheme, EXCMG-Newton method, high efficiency, bi-quartic interpolation.

1. Introduction

Semilinear Poisson equations appear in various fields, including fluid mechanics and geophysics. In this work, we consider a fast solver, which allows us to efficiently obtain numerical solutions of two-dimensional (2D) Poisson equations with nonlinear forcing terms. These equations have the form

$$\begin{aligned} &u_{xx}+u_{yy}=f(u,x,y), && (x,y)\in\Omega,\\ &u(x,y)=g(x,y), && (x,y)\in\partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a rectangle domain with the boundary $\partial\Omega$. The Dirichlet boundary condition is imposed on $\partial\Omega$. Besides, the nonlinear forcing function $f(u,x,y)$, the boundary function

*Corresponding author. *Email address:* `kejiapan@csu.edu.cn` (K.J. Pan)

$g(x, y)$ and the true solution $u(x, y)$ are supposed to be continuously differentiable and have indispensable partial derivatives. Following [19, 33], we assume that the problem (1.1) has a unique solution.

High-order compact finite difference (FD) schemes for Poisson equation have been widely studied [17, 19, 30, 34, 35]. The methods are called compact because the corresponding discretization formulas use a minimal number of mesh points in order to achieve the fourth-order accuracy. However, for large-scale problems, general iteration solvers are quite time-consuming. The multigrid method [3] is a very efficient strategy for solving large sparse systems of linear equations arising in these discretizations. Therefore, the combinations of multigrid methods and high-order compact FD schemes have been also developed [11, 13, 30, 35]. The classical multigrid technique has been applied to other equations — e.g. to convection-diffusion equation [12, 14, 31], the biharmonic equation [1] and the Helmholtz equation [10, 28].

The CMG technique developed by Deuflhard and Bornemann [2], represents an one-way multigrid method without any correlation between fine and coarse grids. The method initially used a conjugate gradient (CG) solver as the relax operator on the embedded grids, whereas the initial values of the smoother on the current grids have been approximated by the linear interpolation on the previous grids. For the energy norm, Bornemann and Deuflhard [2] showed that this is an optimal iterative method. We note that the Richardson extrapolation often used to increase the accuracy of numerical solutions, have been initially employed the coarse grids only. In 1993, Roache and Knupp [25] generalized the strategy and obtained extrapolated solutions at the middle of the fine grid points. This is similar to the mid-point extrapolation formula proposed by Chen and Lin [5].

In the past decade, a CMG method has been combined with extrapolation strategies. Thus Chen *et al.* [4] developed an EXCMG method for fast solution of second-order elliptic problems. Besides, an EXCMG method employing high-order compact FD schemes for 2D Poisson equations have been studied in [6, 15, 21]. Pan *et al.* [22] applied the EXCMG method to 3D elliptic boundary value problems. In order to reduce computational time, Dai *et al.* [7] developed an approximation method, where EXCMG has been used for finding a better initial guess in the MSMG method. Recently, Pan *et al.* [23, 24] applied a fast cell-centered EXCMG (CEXCMG) algorithm based on finite volume (FV) discretization to 2D/3D anisotropic diffusion equations with discontinuous coefficients.

Although for linear problems the EXCMG methods are thoroughly studied, there are only a few works devoted to nonlinear problems. As far as the MG method is concerned, the investigations are mainly focused on Newton-MG methods [3], adaptive MG methods [32] and CMG methods [27, 29, 33]. In particular, for solving large nonlinear systems arising in the fourth-order compact FD discretization of the 2D semilinear Poisson equation, Li and Pan [16] proposed a Newton-MSMG method such that the corresponding extrapolation solutions can achieve the sixth-order approximation accuracy, which is much greater than the discretization-level.

The main purpose of this work is to construct and analyze a fast EXCMG-Newton method for the nonlinear system arising in fourth-order compact FD schemes for the 2D semilinear Poisson equation. Using the extrapolation and bi-quartic polynomial interpolation on