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Global Well-Posedness and Asymptotic Behavior for the 2D Subcritical Dissipative Quasi-Geostrophic Equation in Critical Fourier-Besov-Morrey Spaces

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Abstract. In this paper, we study the subcritical dissipative quasi-geostrophic equation. By using the Littlewood Paley theory, Fourier analysis and standard techniques we prove that there exists v a unique global-in-time solution for small initial data belonging to the critical Fourier-Besov-Morrey spaces $\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}$. Moreover, we show the asymptotic behavior of the global solution v. i.e., $\|v(t)\|_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}}$ decays to zero as time goes to infinity.

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1 Introduction

In this article, we consider the following Cauchy problem for the two-dimensional quasi-geostrophic equation (2DGQ) with subcritical dissipation $\alpha > 1/2$.

$$\begin{cases} \partial_t v + k\Lambda^{2\alpha} v + u_v \cdot \nabla v = 0, \quad x \in \mathbb{R}^2,\ t > 0, \\ u_v = (-\Re_2 v, \Re_1 v), \\ v(0,x) = v_0(x), \end{cases} \tag{1.1}$$

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where $\Re_j=\partial_{x_j}(-\Delta)^{-1/2}$, $j=1,2$, are the Riesz transforms, $\alpha>1/2$ is a real number, $k>0$ is a dissipative coefficient $\big($when $k=0$ Eq. (1.1) becomes the two-dimensional non-dissipative quasi-geostrophic equation $\big)$. Notice that (1.1) is called **subcritical** when $\alpha>1/2$, **critical** when $\alpha=1/2$ and **supercritical** when $\alpha<1/2$. Λ is the operator defined by the fractional power of $-\Delta$:

$$\Lambda v=(-\Delta)^{\frac{1}{2}}v, \quad \mathcal{F}(\Lambda v)=\mathcal{F}((-\Delta)^{\frac{1}{2}}v)=|\xi|\mathcal{F}(v),$$

and more generally

$$\mathcal{F}(\Lambda^{2\alpha}v)=\mathcal{F}((-\Delta)^{\alpha}v)=|\xi|^{2\alpha}\mathcal{F}(v),$$

where $\mathcal{F}$ is the Fourier transform. The scalar function $v(x,t)$ represents the potential temperature, and u_v is the divergence free velocity which is determined by the Riesz transformation of v. Since we are concerned with the dissipative case, we assume $k=1$ for the sake of simplicity.

The 2D quasi-geostrophic equation is an important model in geophysical fluid dynamics, which represents the potential temperature dynamics of atmospheric and ocean flow. For further information on the physical background of this equation, see [1, 2] and the references therein. It is well known that Eq. (1.1) is comparable to the three-dimensional Navier-Stokes equations (see [3–5]).

There is a rich literature about global-in-time well-posedness for fluid dynamics PDEs in different spaces, where the smallness conditions are taken in norms of critical spaces (i.e., the norm is invariant under the scaling of the equation/system). For instance, for Navier-Stokes equations, 2D quasi-geostrophic equations, and related models, we have well-posedness results in the critical case of the following spaces: Lebesgue space L^p [6,7], Marcinkiewicz space $L^{p,\infty}$ [8, 9], Morrey spaces $\mathcal{M}_{p,\mu}$ [10], Besov-Morrey spaces $\mathcal{N}^s_{p,\mu,q}$ [11], Fourier-Besov spaces $\mathcal{FB}^s_{p,q}$ [5, 12], Fourier-Herz spaces $\mathcal{FB}^s_{1,q}=\mathcal{B}^s_{1,q}$ [13], Fourier-Besov-Morrey spaces $\mathcal{FN}^s_{p,\mu,q}$ [14–19], and BMO^{-1} [20], among others. Moreover, in some of the above references, one can find results on decay and/or asymptotic behavior of solutions, such as the works [6–11, 19].

Now, we recall the scaling property of the equations:
if v solves (1.1) with initial data v_0, then v_γ with $v_\gamma(x,t):=\gamma^{2\alpha-1}v\big(\gamma x,\gamma^{2\alpha}t\big)$ is also a solution to (1.1) with the initial data

$$v_{0,\gamma}(x):=\gamma^{2\alpha-1}v_0(\gamma x). \tag{1.2}$$

Definition 1.1. *A critical space for initial data of Eq.* (1.1) *is any Banach space* $E\subset\mathcal{S}'(\mathbb{R}^n)$ *whose norm is invariant under the scaling* (1.2) *for all* $\gamma>0$*, i.e.*

$$\|v_{0,\gamma}(x)\|_E\approx\|v_0(x)\|_E.$$

In accordance with these scales, we can show that the space $\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}$ is critical for (1.1). In this respect, there are several papers on global-in-time well-posedness for