Journal of Computational Mathematics
Vol.41, No.1, 2023, 39–71.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2104-m2020-0231

RECONSTRUCTED DISCONTINUOUS APPROXIMATION TO STOKES EQUATION IN A SEQUENTIAL LEAST SQUARES FORMULATION*

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Abstract

We propose a new least squares finite element method to solve the Stokes problem with two sequential steps. The approximation spaces are constructed by the patch reconstruction with one unknown per element. For the first step, we reconstruct an approximation space consisting of piecewise curl-free polynomials with zero trace. By this space, we minimize a least squares functional to obtain the numerical approximations to the gradient of the velocity and the pressure. In the second step, we minimize another least squares functional to give the solution to the velocity in the reconstructed piecewise divergence-free space. We derive error estimates for all unknowns under both L^2 norms and energy norms. Numerical results in two dimensions and three dimensions verify the convergence rates and demonstrate the great flexibility of our method.

Mathematics subject classification: 65N30.
Key words: Stokes problem, Least squares finite element method, Reconstructed discontinuous approximation, Solenoid and irrotational polynomial bases.

1. Introduction

The Stokes problem, which models a viscous and incompressible fluid flow, is a linearized version of the full Navier-Stokes equation neglecting the nonlinear convective term. Reliable and efficient numerical methods for the Stokes problem have been extensively studied in the references. Among these methods, there were many efforts devoted to develop mixed finite element methods based on the weak formulation of the Stokes problem. A key issue of classical mixed finite element methods is the choice of element types. The pair of finite element spaces are required to satisfy the stability condition, known as the inf-sup condition. We refer the readers to [10, 12, 18] for some examples in classical mixed finite element methods.

The least squares finite element methods for the Stokes problem have been developed in [5–8, 16, 20, 28, 31]. For these methods, least squares principle together with finite element methods can offer the advantage of circumventing the inf-sup condition arising in mixed methods. Bochev and Gunzburger developed a least squares approach based on rewriting the velocity-vorticity-pressure formulation as a first-order elliptic system [8]. Cai and his coworkers developed the least squares finite element method based on the L^2 norm residual and C^0 spaces for the Stokes

* Received September 1, 2020 / Revised version received April 13, 2021 / Accepted April 26, 2021 / Published online February 26, 2022 /
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problem, we refer to [5, 14–16] for more details. Liu et al. developed a hybrid least squares finite element method based on continuous finite element spaces. This method attempts to combine the advantages of FOSLS and FOSLL* [28]. The works introduced above are based on conforming finite element spaces and such continuous least squares methods are general techniques in numerical methods. We refer to [9] and the references therein for an overview of least squares finite element methods. Based on discontinuous approximation, the discontinuous least squares finite element methods have also been developed for many problems including the Stokes problem, and we refer to [3, 4, 6, 7, 17, 26] for more details.

In this paper, we propose a new least squares finite element method with the reconstructed discontinuous approximation. The novelty is that we construct three specific approximation spaces which allow us to solve the Stokes problem in two sequential steps. The sequential process is motivated from the idea in [16, 27] to define two least-squares-type functionals to approximate unknowns sequentially. The feasibility of this method is based on new approximation spaces which are obtained by solving local least squares problems on each element. In the first step, we reconstruct an approximation space that consists of piecewise irrotational polynomials with zero trace to approximate the gradient of the velocity. This space is an extension of the space proposed in [24], which will also be used in this step to approximate the pressure. The functions in both approximation spaces may be discontinuous across interior faces and we define a least squares functional with the weak imposition of the continuity across the interior faces to seek numerical solutions in approximation to the gradient and the pressure. In the second step, we reconstruct a piecewise divergence-free polynomial space to approximate the velocity, which is also a generalization of the space in [24]. We minimize another least squares functional, together with the numerical gradient obtained in the first step, to solve the numerical solution for the velocity. For the error estimate, we introduce a series of projection operators to derive the convergence rates for all variables with respect to L^2 norms and energy norms. We prove that the convergence orders under energy norms are all optimal and the L^2 errors for all variables can only be proved to be sub-optimal. We conduct a series of numerical examples in two dimensions and three dimensions to confirm our theoretical error estimates. In addition, we observe that the L^2 errors for all unknowns are optimally convergent for approximation spaces of odd orders. Another advantage of our method is the implementation is quite simple. The different types of the reconstruction can be implemented in a uniform way. We present the details to the computer implementation of our method in Appendix 6.

The rest of our paper is organized as follows. In Section 2, we give the notation that will be used in this paper. In Section 3, we introduce the reconstruction operators and the corresponding approximation spaces. The approximation properties of spaces are also presented in this section. In Section 4, we define two least squares functionals for sequentially solving the Stokes problem with reconstructed spaces. We also prove the error estimates under L^2 norms and energy norms. In Section 5, we present a series of numerical results in two dimensions and three dimensions to illustrate the accuracy and the flexibility of our method.

2. Preliminaries

We let Ω be a convex bounded polygonal (polyhedral) domain in $\mathbb{R}^d (d = 2, 3)$ with the boundary $\partial\Omega$. We denote by $\mathcal{T}_h$ a set of polygonal (polyhedral) elements which partition the domain Ω. We denote by $\mathcal{E}_h^i$ the set of interior faces of $\mathcal{T}_h$ and by $\mathcal{E}_h^b$ the set of the faces that are on the boundary $\partial\Omega$. Let $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$ be the set of all $d-1$ dimensional faces. Further,