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# REQUIRED NUMBER OF ITERATIONS FOR SPARSE SIGNAL RECOVERY VIA ORTHOGONAL LEAST SQUARES*

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**Abstract**

In countless applications, we need to reconstruct a $K$-sparse signal $\mathbf{x} \in \mathbb{R}^n$ from noisy measurements $\mathbf{y} = \mathbf{\Phi x} + \mathbf{v}$, where $\mathbf{\Phi} \in \mathbb{R}^{m\times n}$ is a sensing matrix and $\mathbf{v} \in \mathbb{R}^m$ is a noise vector. Orthogonal least squares (OLS), which selects at each step the column that results in the most significant decrease in the residual power, is one of the most popular sparse recovery algorithms. In this paper, we investigate the number of iterations required for recovering $\mathbf{x}$ with the OLS algorithm. We show that OLS provides a stable reconstruction of all $K$-sparse signals $\mathbf{x}$ in $\lceil 2.8K\rceil$ iterations provided that $\mathbf{\Phi}$ satisfies the restricted isometry property (RIP). Our result provides a better recovery bound and fewer number of required iterations than those proposed by Foucart in 2013.

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*Key words:* Sparse signal recovery, Orthogonal least squares (OLS), Restricted isometry property (RIP).

## 1. Introduction

Compressed sensing (CS) has been attracted considerable attention in numerous fields [1–5]. The main task of CS is to recover a signal $\mathbf{x} \in \mathbb{R}^n$ from

$$\mathbf{y} = \mathbf{\Phi x} + \mathbf{v}, \tag{1.1}$$

where $\mathbf{\Phi} \in \mathbb{R}^{m\times n}(m \ll n)$ is a sensing matrix with $\ell_2$-normalized columns, $\mathbf{x}$ is a $K$-sparse (i.e., $\|\mathbf{x}\|_0 \le K$, where $\|\mathbf{x}\|_0$ denotes the number of nonzero entries of $\mathbf{x}$) signal, and $\mathbf{v} \in \mathbb{R}^m$ is a noise vector.

There are many algorithms ([6–12]) for recovering $\mathbf{x}$ from (1.1). One of the popular one is the orthogonal least squares (OLS) [13–16] algorithm. It has been shown in [15] that OLS

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is computationally more expensive yet is more reliable than the orthogonal matching pursuit (OMP) algorithm [17, 18], hence it has been attracted much attention in recent years. OLS identifies the support of $\mathbf{x}$ by adding one index to the list at each iteration, and estimates the coefficients of the sparse vector over the enlarged support. Specifically, it adds to the estimated support an index which leads to the maximum reduction of the residual power in each iteration. The vestige of the active list is then eliminated from $\mathbf{y}$, yields a residual update for the next iteration.

Denote $\Omega = \{1, \cdots, n\}$ and $T = \text{supp}(\mathbf{x}) = \{i | x_i \neq 0, i \in \Omega\}$ as the support of $K$-sparse signal $\mathbf{x}$. Let $\Lambda$ be a subset of $\Omega$, $|\Lambda|$ be the cardinality of $\Lambda$, and $T \backslash \Lambda = \{i | i \in T, i \notin \Lambda\}$. Let $\mathbf{x}_\Lambda \in \mathbb{R}^n$ be the vector equal to $\mathbf{x}$ on the index set $\Lambda$ and zero elsewhere. Throughout the paper, we assume that $\mathbf{\Phi} \in \mathbb{R}^{m \times n}$ is column normalized (i.e., $\|\mathbf{\Phi}_i\|_2 = 1$ for $i = 1, 2, \cdots, n$)[1)] . Let $\mathbf{\Phi}_\Lambda \in \mathbb{R}^{m \times |\Lambda|}$ be the submatrix of $\mathbf{\Phi}$ with index of its columns in set $\Lambda$. For any matrix $\mathbf{\Phi}_\Lambda$ of full column-rank, let $\mathbf{\Phi}_\Lambda^\dagger = (\mathbf{\Phi}'_\Lambda \mathbf{\Phi}_\Lambda)^{-1} \mathbf{\Phi}'_\Lambda$ be the pseudo-inverse of $\mathbf{\Phi}_\Lambda$, where $\mathbf{\Phi}'_\Lambda$ denotes the transpose of $\mathbf{\Phi}_\Lambda$. $\mathbf{P}_\Lambda = \mathbf{\Phi}_\Lambda \mathbf{\Phi}_\Lambda^\dagger$ and $\mathbf{P}_\Lambda^\perp = \mathbf{I} - \mathbf{P}_\Lambda$ denote the orthogonal projection onto span($\mathbf{\Phi}_\Lambda$) (i.e., the column space of $\mathbf{\Phi}_\Lambda$) and its orthogonal complement, respectively. OLS is mathematically described in Algorithm 1.1.

**Algorithm 1.1.** The OLS algorithm [19]

**Input:** $\mathbf{\Phi}$, $\mathbf{y}$, maximum iteration number $k_{\max}$.
**Initialization:** For $\mathbf{r}^0 = \mathbf{y}$, $k = 0$, and $S^0 = \emptyset$.
1: **while** $k < k_{\max}$ **do**
2: $k = k + 1$. 3: Choose the index $s^k$ that satisfies
$s^k = \arg\min_{i \in \Omega} \|\mathbf{P}^\perp_{S^{k-1} \cup \{i\}} \mathbf{y}\|_2^2$.
4: Let $S^k = S^{k-1} \bigcup \{s^k\}$, and calculate
$\mathbf{x}^k = \arg\min_{\text{supp}(\mathbf{u}) = S^k} \|\mathbf{y} - \mathbf{\Phi}\mathbf{u}\|_2$.
5: $\mathbf{r}^k = \mathbf{y} - \mathbf{\Phi}\mathbf{x}^k = \mathbf{P}^\perp_{S^k} \mathbf{y}$.
6: **end while**
**Output:** $\mathbf{x}^k$ and $S^k$.

The performance analysis of OLS has been extensively studied. For example, Soussen *et al.* showed that OLS is guaranteed to exactly recover the support of $\mathbf{x}$ in at most $K$ iterations when the exact recovery condition (ERC) is met [14]. Based on mutual coherence, Herzet *et al.* addressed the exact recovery of $\mathbf{x}$ in the noiseless setting when some partial information of its support is available [15]. Herzet *et al.* developed extended coherence-based sufficient conditions for exact sparse support recovery with OLS [16]. Wen *et al.* [19] and Geng *et al.* [20] utilized the restricted isometry property (RIP), which is defined as follows, to study the sufficient condition of exact recovery of $\mathbf{x}$ with OLS. Using the RIP, the authors in [21–24] discussed the performance of multiple OLS which is an extension of OLS.

**Definition 1.1 ([25]).** *A measurement matrix* $\mathbf{\Phi}$ *is said to satisfy the RIP of order* $K$ *if there exists a constant* $\delta \in [0, 1)$ *such that,*

$$(1 - \delta)\|\mathbf{x}\|_2^2 \leq \|\mathbf{\Phi}\mathbf{x}\|_2^2 \leq (1 + \delta)\|\mathbf{x}\|_2^2 \tag{1.2}$$

[1)] The behavior of OLS is unchanged whether columns of $\mathbf{\Phi}$ are normalized or not ([31]).