Commun. Math. Res.
doi: 10.4208/cmr.2021-0096

Vol. **38**, No. 4, pp. 516-534
November 2022

Subelliptic Harmonic Maps with Values in Metric Spaces of Nonpositive Curvature

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Received 19 November 2021; Accepted 6 May 2022

Abstract. We prove the Hölder continuity of a harmonic map from a domain of a sub-Riemannian manifold into a locally compact manifold with nonpositive curvature, and more generally into a non-positively curved metric space in the Alexandrov sense.

AMS subject classifications: 51F99, 31E05

Key words: Subelliptic harmonic maps, nonpositive curvature, Hölder continuity.

1 Introduction

Sub-Riemannian manifolds naturally arise in many areas of mathematics. They are generalizations of Riemannian manifolds, where the quadratic form on the tangent bundle no longer is strictly positive definite, but where still all directions can be generated from commutators of positive ones. Thus, a sub-Riemannian manifold is a connected smooth manifold equipped with a positive definite quadratic form Q defined on a smooth distribution S with rank m of the tangent bundle, and S is assumed to satisfy the Hörmander condition. This means that the vector fields in S together with their brackets up to some order generate the whole tangent bundle. Thus, while we no longer require positive definiteness, as discovered by Hörmander [6] this condition still implies some useful analysis inequalities, and with some additional effort, one can usually derive the same results as

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are known for Riemannian manifolds. To see the structure, let us first point out that for any point $x \in M$, one can use the quadratic Q_x to define a unique linear mapping $g_S(x): T_x^* \to T_x$ via

$$Q_x\big(g_S(x)\xi, X\big) = \langle \xi, X \rangle, \quad \forall X \in S_x$$

for $\xi \in T_x^*$, where $\langle , \rangle$ is the pairing between T_x^* and T_x. g_S varies smoothly on M, and is semi positive definite.

There is one difficulty in sub-Riemannian geometry, however. Since the metric is not positive definite, there is no canonically defined volume measure. To overcome this problem, we choose an arbitrary Riemannian metric g and we equip the bundle complementary to the distribution S with the restriction of the metric g and we then declare it to be orthogonal to S. We then obtain a measure and a volume form, although admittedly the choice of the metric g is arbitrary and not determined by the sub-Riemannian structure.

On the positive side, a distance function is naturally associated with the distribution. In fact, for any given two points, by a classical theorem of Chow and Rashevsky, see [2,15], because of the Hörmander condition, we can find a curve connecting them whose tangent vectors are contained in the distribution S and therefore have positive norm, and in particular, the curve then has positive length. We can then define the distance by taking the infimum of the lengths of such curves. Hence, the distance is finite, and we obtain a metric. In particular, we can find shortest curves, geodesics, connecting any two points.

Dong [3] showed the existence of harmonic maps from sub-Riemannian manifolds into Riemannian manifolds of non-positive curvature. Here, in order to define the energy functional, the above volume form is needed. In this contribution, we show the regularity of such harmonic maps. Our main theorem is

Theorem 1.1. *Any harmonic map $u: M \to N$ from a sub-Riemannian manifold M into a locally compact Riemannian manifold N with non-positive curvature is Hölder continuous.*

In fact, our proofs make no use of the smooth structure of the target, hence our method applies to a more general setting, and we can prove

Theorem 1.2. *Let $u: M \to N$ be a harmonic map from a sub-Riemannian manifold M into a locally compact metric space N which is non-positively curved in the Alexandrov sense. Then u is Hölder continuous.*

As mentioned, the existence of the harmonic map has been obtained in [3] when the target is smooth with non-positive curvature. The method can be extended to more general metric spaces of non-positive curvature, for instance by using the constructions of [8].