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Numerical Solution of Partial Differential Equations in Arbitrary Shaped Domains Using Cartesian Cut-Stencil Finite Difference Method. Part II: Higher-Order Schemes

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Abstract. Compact higher-order (HO) schemes for a new finite difference method, referred to as the Cartesian cut-stencil FD method, for the numerical solution of the convection-diffusion equation in complex shaped domains have been addressed in this paper. The Cartesian cut-stencil FD method, which employs 1-D quadratic transformation functions to map a non-uniform (uncut or cut) physical stencil to a uniform computational stencil, can be combined with compact HO Padé-Hermitian formulations to produce HO cut-stencil schemes. The modified partial differential equation technique is used to develop formulas for the local truncation error for the cut-stencil HO formulations. The effect of various HO approximations for Neumann boundary conditions on the solution accuracy and global order of convergence are discussed. The numerical results for second-order and compact HO formulations of the Cartesian cut-stencil FD method have been compared for test problems using the method of manufactured solutions.

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Key words: Cartesian cut-stencil finite difference method, compact higher-order formulation, irregular domain, Neumann boundary conditions, local truncation error.

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1. Introduction

In finite difference methods, the truncation (or discretization) error TE is a mea-

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sure of the accuracy of the approximate numerical solution of a partial differential equation (PDE). The TE determines the order of accuracy (p) by expressing $\mathrm{TE} \approx h^p$, where h denotes the grid step size [19]. Higher-order (higher than second-order) approximations, formulations and related algorithms have gained intense interest in different application areas of science and engineering such as compressible and incompressible flow, computational aeroacoustics, geodynamic simulations and aerospace [11, 23, 29, 38, 39, 52].

Higher-order (HO) schemes have also been implemented in conjunction with the finite volume method (FVM), mostly employing wider stencils to calculate the fluxes at the cell interfaces. However, HO schemes are more suitable and straightforward for structured grids [5], in contrast to the main advantage of FVM, which can handle complex domains and unstructured meshes. Furthermore, although some higher-order FVM-based formulations have been constructed for unstructured grids [4, 21, 49], they are not exempt from difficulty because of very complicated algorithms and implementation procedures. Meanwhile, the total variation diminishing (TVD) method proposed by Harten [20] shows first-order accurate behavior that suppresses numerical oscillations. The higher-order accurate essentially non-oscillatory (ENO) scheme was developed by Harten *et al.* [22] to overcome the issue of TVD schemes that require smooth computational grids. Again, implementation of these schemes may be difficult, if not impossible, to achieve for complex domains [49]. Other HO accurate schemes for arbitrary domains, such as the high-order k-exact finite volume scheme [4], are known to suffer inefficient use of computational memory [49]. There are also many numerical studies devoted to the construction of various techniques for higher-order finite element methods (FEM) [34]. Generally speaking, in FEM, higher-order refers to the order of the elements (i.e., degree of the polynomial approximation of the integrand) used in the formulation.

The finite difference method (FDM), which is based on Taylor series approximation for the derivatives in the governing equations, is regarded by most researchers as the simplest numerical discretization method. However, traditionally, FDM can be directly applied only on rectangular shaped domains in which grid points can be equally spaced [33]. FDM encounters serious difficulties for complex domains, particularly at nodes near boundaries of the physical domain [25]. Numerical techniques such as structured body-fitted curvilinear grids and multiblock methods have been developed as a remedy to allow application of FDM to solve partial differential equations in complex domains [6, 15, 16, 31, 44]. The process of designing a good quality body-fitted grid can be very labour-intensive and may be impractical from the perspective of overall computational cost and time. Similarly, the multiblock technique is somewhat difficult to implement since it needs considerable experience to generate a good quality grid system for irregular shaped domains.

The Cartesian cut-stencil FDM (CCST-FDM) is capable of solving PDEs in irregular shaped domains. The fundamental details of this finite difference formulation have been presented in Esmaeilzadeh [12] and Esmaeilzadeh *et al.* [13]. This method, in its basic formulation, employs first- and second-order differencing schemes to approx-