Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.OA-2022-0015

Vol. **15**, No. 3, pp. 793-818
August 2022

The Recovery Guarantee for Orthogonal Matching Pursuit Method to Reconstruct Sparse Polynomials

Aitong Huang, Renzhong Feng* and Sanpeng Zheng

School of Mathematics Science & Key Laboratory of Mathematics, Informatics and Behavioral Semantics, Ministry of Education, Beihang University, Beijing, China

Received 24 January 2022; Accepted (in revised version) 10 May 2022

Abstract. Orthogonal matching pursuit (OMP for short) algorithm is a popular method of sparse signal recovery in compressed sensing. This paper applies OMP to the sparse polynomial reconstruction problem. Distinguishing from classical research methods using mutual coherence or restricted isometry property of the measurement matrix, the recovery guarantee and the success probability of OMP are obtained directly by the greedy selection ratio and the probability theory. The results show that the failure probability of OMP given in this paper is exponential small with respect to the number of sampling points. In addition, the recovery guarantee of OMP obtained through classical methods is lager than that of ℓ_1-minimization whatever the sparsity of sparse polynomials is, while the recovery guarantee given in this paper is roughly the same as that of ℓ_1-minimization when the sparsity is less than 93. Finally, the numerical experiments verify the availability of the theoretical results.

AMS subject classifications: 41A10, 41A05, 65D15

Key words: Reconstruction of sparse polynomial, uniformly bounded orthogonal system, orthogonal matching pursuit method, probability of successful reconstruction, sub-Gaussian random variable.

1. Introduction

The reconstruction of sparse polynomials is a popular research topic in the field of approximation theory in recent years [3, 29]. Suppose that $g(x)$ is in the form of

$$g(x) = \sum_{j\in\Lambda} c_j \phi_j(x),$$

*Corresponding author. *Email address:* fengrz@buaa.edu.cn (R. Feng)

where $\{\phi_j(x)\}_{j\in\Lambda}$ is a set of basis functions defined on $\Omega \subset \mathbb{C}^d$, and Λ is an index set with $|\Lambda| = n$, where n can be finite or infinite, $|\cdot|$ represents the number of elements in the set. Let the vector $\mathbf{c} = [c_1, \ldots, c_n]^\top \in \mathbb{C}^n$ be the coefficient vector composed of coefficients of $g(x)$. If the coefficient vector $\mathbf{c}$ has at most s elements that are not 0, here $1 \le s \ll n$, the polynomial $g(x)$ is called an s-sparse polynomial, and s is called the sparsity of the sparse polynomial $g(x)$ and the sparse vector $\mathbf{c}$. Therefore, the problem of reconstructing the sparse polynomial $g(x)$ can be transformed into reconstructing the sparse vector $\mathbf{c}$. In this paper, we study the recovery problem when the dimension $d = 1$, n is finite and the basis functions $\{\phi_j(x)\}_{j\in\Lambda}$ is a uniformly bounded orthonormal system.

1.1. Introduction to compressed sensing

In recent years, compressed sensing has been developed rapidly [1, 9, 26, 28]. Its main idea is to use nonlinear optimization to recover a sparse signal by as few observations as possible. The original model for sparse signal recovery is

$$\min_{\mathbf{c}\in\mathbb{C}^n} \|\mathbf{c}\|_0 \quad \text{s.t.} \quad \Phi\mathbf{c} = \mathbf{b}, \tag{1.1}$$

where $\|\mathbf{c}\|_0$ represents the number of non-zero elements in the vector $\mathbf{c}, \Phi \in \mathbb{C}^{m\times n}$ is a measurement matrix, $\mathbf{b} = [b_1, \ldots, b_m]^\top \in \mathbb{C}^{m\times 1}$ is an observation vector. Unfortunately, the model (1.1) is an NP-hard problem. Therefore, many scholars have considered whether there are other ways to recover the sparse signal $\mathbf{c}$ [12]. One way is to convert (1.1) into

$$\min_{\mathbf{c}\in\mathbb{C}^n} \|\mathbf{c}\|_1 \quad \text{s.t.} \quad \Phi\mathbf{c} = \mathbf{b}, \tag{1.2}$$

where $\|\mathbf{c}\|_1 = \sum_{j\in\Lambda} |c_j|$. It is proved that when the measurement matrix Φ satisfies the null space property, (1.1) and (1.2) are equivalent [11]. Many iterative algorithms are designed to solve (1.2), such as Bregman iterative algorithm [16] and ADM (alternating direction method) [31]. Another way is greedy algorithm [17, 23]. If we know in advance that the sparsity of the signal to be reconstructed is s, then we convert the model (1.1) into the following ℓ_2-norm model with inequality constraints:

$$\min_{\mathbf{c}\in\mathbb{C}^n} \|\Phi\mathbf{c} - \mathbf{b}\|_2 \quad \text{s.t.} \quad \|\mathbf{c}\|_0 \le s. \tag{1.3}$$

At present, many researchers have given a lot of greedy algorithms to solve model (1.3) [5, 6, 8, 23, 25, 32], among which orthogonal matching pursuit algorithm (OMP for short) is the most important method [17].

1.2. Sparse polynomial reconstruction via ℓ_1-minimization

The problem of using ℓ_1-minimization to reconstruct sparse polynomials has been considered for many years [19, 21, 29, 30], of which the sampling method and recovery guarantee are the most two popular topics. In 2006, Candes *et al.* [1, 2] chose sampling