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On Energy Stable Runge-Kutta Methods for the Water Wave Equation and its Simplified Non-Local Hyperbolic Model

Lei Li[1], Jian-Guo Liu[2], Zibu Liu[3,*], Yi Yang[4] and Zhennan Zhou[5]

[1] *School of Mathematical Sciences, Institute of Natural Sciences, MOE-LSC, Shanghai Jiao Tong University, Shanghai, 200240, P.R. China.*
[2] *Department of Mathematics and Department of Physics, Duke University, Durham, NC 27708, USA.*
[3] *Department of Mathematics, Duke University, Durham, NC 27708, USA.*
[4] *Nanjing Research Institute of Electronics Technology, Nanjing, P.R. China.*
[5] *Beijing International Center for Mathematical Research, Peking University, P.R. China.*

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Abstract. Although interest in numerical approximations of the water wave equation grows in recent years, the lack of rigorous analysis of its time discretization inhibits the design of more efficient algorithms. In practice of water wave simulations, the trade-off between efficiency and stability has been a challenging problem. Thus to shed light on the stability condition for simulations of water waves, we focus on a model simplified from the water wave equation of infinite depth. This model preserves two main properties of the water wave equation: non-locality and hyperbolicity. For the constant coefficient case, we conduct systematic stability studies of the fully discrete approximation of such systems with the Fourier spectral approximation in space and general Runge-Kutta methods in time. As a result, an optimal time discretization strategy is provided in the form of a modified CFL condition, i.e. $\Delta t = \mathcal{O}(\sqrt{\Delta x})$. Meanwhile, the energy stable property is established for certain explicit Runge-Kutta methods. This CFL condition solves the problem of efficiency and stability: it allows numerical schemes to stay stable while resolves oscillations at the lowest requirement, which only produces acceptable computational load. In the variable coefficient case, the convergence of the semi-discrete approximation of it is presented, which naturally connects to the water wave equation. Analogue of these results for the water wave equation of finite depth is also discussed. To validate these theoretic observation, extensive numerical tests have been performed to verify the stability conditions. Simulations of the simplified hyperbolic model in the high frequency regime and the water wave equation are also provided.

*Corresponding author. *Email addresses:* `leili2010@sjtu.edu.cn` (L. Li), `jliu@phy.duke.edu` (J.-G. Liu), `zibu.liu@duke.edu` (Z. Liu), `sailors2008@sina.cn` (Y. Yang), `zhennan@bicmr.pku.edu.cn` (Z. Zhou)

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1 Introduction

Simulation of water wave equations has been a challenging problem due to the bad well-posedness of the equation. To understand this difficulty, we will first review previous remarkable theoretic work on water wave equations. In both Wu's work [28] and Beale and Hou's work [5], the authors used Riemann mappings to find the right variables and rewrote the water wave equation. Both versions of the equation in [28] and [5] have the common leading order structure: a non-local hyperbolic equation, say (1.6). By analyzing this simplified model, we derive an optimal discretization strategy in the form of a CFL condition. This condition is rigorously proved for system (1.6) and numerically verified for water wave equations. Following this road map, we will first derive the simplified model.

The unsteady system of incompressible free surface flow in two-dimension has attracted much theoretic and numerical attention [6, 10, 11, 28]. Governed by the irrotational Euler equation, this free surface flow problem is also referred to as the water wave problem which dates back to the early 20th century [23, 25]. By observing the equation in both Eulerian coordinate and Lagrangian coordinate, insightful analytical results were derived since then. One can refer to [11] for a review of recent related results. In [5], Beale, Hou, and Lowengrub formulated the water wave equation in Lagrangian coordinates and considered the linearization of it which is a nonlocal system. Later in [28], by reducing the system to a nonlocal hyperbolic equation in Eulerian coordinate, Wu derived impressive results on the well-posedness of the water wave equation. These two works directed us to focus on a simplified system that inherits the common dominating structure shared by both works: a nonlocal hyperbolic model, which played a critical role in the proof of well-posedness in [28].

The water wave equation is formulated as follows (see [5]) in Lagrangian coordinates. Consider a 2π-periodic two-dimensional fluid with infinite depth whose surface is described by $z: \mathbb{R}\times[0,\infty)\to\mathbb{C}$:

$$z(\alpha,t)=x(\alpha,t)+iy(\alpha,t). \tag{1.1}$$

Here $\alpha\in\mathbb{R}$ is a material coordinate that parametrizes the undisturbed surface. Periodicity of the fluid wave implies that $s(\alpha,t):=z(\alpha,t)-\alpha$ is a 2π-periodic function in α. Because the fluid is inviscid and irrotational, the velocity can be written as $\nabla\Phi$ where $\Phi(x,y,t)$ is the velocity potential. Let

$$\phi:\ \mathbb{R}\times[0,\infty)\to\mathbb{R},\quad (\alpha,t)\mapsto\phi(\alpha,t):=\Phi(x(\alpha,t),y(\alpha,t),t)$$