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Global Well-Posedness of Solutions to 2D Prandtl-Hartmann Equations in Analytic Framework

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Abstract. In this paper, we consider the two-dimensional (2D) Prandtl-Hartmann equations on the half plane and prove the global existence and uniqueness of solutions to 2D Prandtl-Hartmann equations by using the classical energy methods in analytic framework. We prove that the lifespan of the solutions to 2D Prandtl-Hartmann equations can be extended up to T_ε (see Theorem 2.1) when the strength of the perturbation is of the order of ε. The difficulty of solving the Prandtl-Hartmann equations in the analytic framework is the loss of x-derivative in the term $v\partial_y u$. To overcome this difficulty, we introduce the Gaussian weighted Poincar*é* inequality (see Lemma 2.3). Compared to the existence and uniqueness of solutions to the classical Prandtl equations where the monotonicity condition of the tangential velocity plays a key role, which is not needed for the 2D Prandtl-Hartmann equations in analytic framework. Besides, the existence and uniqueness of solutions to the 2D MHD boundary layer where the initial tangential magnetic field has a lower bound plays an important role, which is not needed for the 2D Prandtl-Hartmann equations in analytic framework, either.

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1 Introduction

The following Prandtl-Hartmann equations were derived in [1] by the classical two dimensional (2D) incompressible MHD system under a transverse magnetic field on the

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boundary for the parameters of the system in the mixed Prandtl-Hartmann regime. In this paper, we shall investigate the global existence and uniqueness of solutions to the following initial boundary value problem for the 2D Prandtl-Hartmann system on the upper half plane $\mathbb{R}^2_+ = \{(x,y): x\in\mathbb{R},\ y\in\mathbb{R}_+\}$, which reads as

$$\begin{cases} \partial_t u_1 + u_1\partial_x u_1 + u_2\partial_y u_1 - \partial_y^2 u_1 + u_1 = 0, \\ \partial_x u_1 + \partial_y u_2 = 0, \end{cases} \tag{1.1}$$

where the velocity field $(u_1(t,x,y), u_2(t,x,y))$ of the boundary layer is an unknown vector function.

The system (1.1) is subject to the initial data and boundary conditions

$$\begin{cases} u_1(t,x,y)|_{t=0} = u_{10}(x,y), \\ u_1|_{y=0} = u_2|_{y=0} = 0. \end{cases} \tag{1.2}$$

The far field is represented by a positive constant $\bar{u}$, where

$$\lim_{y\to+\infty} u_1(t,x,y) = \bar{u}. \tag{1.3}$$

To start with, let us briefly review some known results to the problem (1.1). Especially, when the damping term u_1 does not exist, the system (1.1) reduces to the classical Prandtl equations which was firstly introduced formally by Prandtl [2] in 1904. This system is the foundation of the boundary layer equations. It describes that the part away from the boundary can be considered as general ideal fluid, but the part near a rigid wall is deeply affected by the viscous force. Formally, the asymptotic limit of the Navier-Stokes equations can be denoted by the Prandtl equations within the boundary layer and by the Euler equations away from boundary. About sixty years later, Oleinik [3] established the first systematic work in strictly mathematic analysis, in which she pointed out that the local-in-time well-posedness of Prandtl system can be proved in 2D by using the Crocco transformation under the monotonicity condition on the tangential velocity field in the normal variable to the boundary. This result together with an expanded introduction to the boundary layer theory was showed in Oleinik-Samokhin's classical book [4].

There have been some results in the analytic framework with analytic radii $\tau(t)$ for 2D boundary layer equations. First, authors [5] investigated the local well-posedness of solutions to the 2D Prandtl and hydrostatic Euler equations by energy methods, which was the original result for boundary layer problem in the analytic framework with analytic radii $\tau(t)$. Kukavica and Vicol [6] considered the local well-posedness of solutions to the 2D Prandtl boundary layer equations with general initial data by using analytic energy estimates in the tangential variables. Later on, Ignatova and Vicol [7] investigated the almost global existence for the two-dimensional Prandtl equations when the