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Mean Field Equations for the Equilibrium Turbulence and Toda Systems on Connected Finite Graphs

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Abstract. In this paper, we study existence of solutions of mean field equations for the equilibrium turbulence and Toda systems on connected finite graphs. Our method is based on calculus of variations, which was built on connected finite graphs by Grigor'yan, Lin and Yang.

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1 Introduction

Let $G=(V,E)$ be a connected finite graph, where V is the vertex set and E the edge set. For every edge $xy\in E$, we assume its weight $w_{xy}>0$ and $w_{xy}=w_{yx}$. Denote by μ a positive and finite measure on V. Let us review some definitions first.

Definition 1.1. *For any function f on V, the μ-Laplacian of f is defined by*

$$\Delta_\mu f(x)=\frac{1}{\mu(x)}\sum_{y\sim x}w_{xy}[f(y)-f(x)],$$

where $y\sim x$ means $xy\in E$. The associated gradient form is

$$\Gamma(f,g)(x)=\frac{1}{2\mu(x)}\sum_{y\sim x}w_{xy}[f(y)-f(x)][g(y)-g(x)].$$

The length of its gradient means

$$|\nabla f|(x)=\sqrt{\Gamma(f,f)(x)}.$$

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Now we are prepared to state our main results.

Theorem 1.1. *Let $G=(V,E)$ be a connected finite graph, μ a positive and finite measure on V. Let $\psi: V\to\mathbb{R}$ be a function with $\int_V \psi \mathrm{d}\mu = 1$. For $i=1,2$, we assume $c_i>0$ is a constant and $0\le h_i \not\equiv 0$ is a function on V. Then the mean field equation for the equilibrium turbulence*

$$\begin{cases} \Delta_\mu u = c_1\left(\psi - \dfrac{h_1 e^u}{\int_V h_1 e^u \mathrm{d}\mu}\right) - c_2\left(\psi - \dfrac{h_2 e^{-u}}{\int_V h_2 e^{-u}\mathrm{d}\mu}\right), \\ \int_V \psi u \mathrm{d}\mu = 0 \end{cases} \tag{1.1}$$

has a solution on V.

Our second result is about Toda system. We shall prove that

Theorem 1.2. *Let $G=(V,E)$ be a connected finite graph, μ a positive and finite measure on V. Let $\psi: V\to\mathbb{R}$ be a function with $\int_V \psi \mathrm{d}\mu = 1$. For $i=1,2$, we assume $\lambda_i>0$ is a constant and $0\le \rho_i \not\equiv 0$ is a function on V. Then the Toda system*

$$\begin{cases} \Delta_\mu u_1 = 2\lambda_1\left(\psi - \dfrac{\rho_1 e^{u_1}}{\int_V \rho_1 e^{u_1}\mathrm{d}\mu}\right) - \lambda_2\left(\psi - \dfrac{\rho_2 e^{u_2}}{\int_V \rho_2 e^{u_2}\mathrm{d}\mu}\right), \\ \Delta_\mu u_2 = 2\lambda_2\left(\psi - \dfrac{\rho_2 e^{u_2}}{\int_V \rho_2 e^{u_2}\mathrm{d}\mu}\right) - \lambda_1\left(\psi - \dfrac{\rho_1 e^{u_1}}{\int_V \rho_1 e^{u_1}\mathrm{d}\mu}\right), \\ \int_V \psi u_i \mathrm{d}\mu = 0, \quad i=1,2 \end{cases} \tag{1.2}$$

has a solution on V.

On a compact Riemann surface (Σ,g), Eq. (1.1) describes the mean field of the equilibrium turbulence with arbitrarily signed vortices [1–3] and was obtained in [4,5] from different statistical arguments. When $\psi=1/|\Sigma|$, Ohtsuka and Suzuki [6], using a variational argument, proved that Eq. (1.1) can be solved if $0\le c_1,c_2<8\pi$ and $h_1=h_2=1$; Zhou [7,8] gave a sufficient condition for the existence of solutions of Eq. (1.1) when $c_1=c_2=8\pi$ and $h_1=h_2=1$; moreover, she studied the supercritical case of the existence of solutions of (1.1). These results partially generalized the existence results about Kazdan-Warner problem in [9].

On a compact Riemann surface (Σ,g), Eq. (1.2) is related to non-Abelian Chern-Simons model [10]. When $\psi=1/|\Sigma|$, Jost-Wang [11] proved the Moser-Trudinger inequality for Toda system. Based on this inequality, Li-Li [12] and Jost-Lin-Wang [13] gave a sufficient condition for the existence of solutions of the Toda system with $\lambda_1=\lambda_2=4\pi$, which is the critical case in the sense of the Moser-Trudinger inequality. For more relevant study, we refer the interesting reader to the references therein.