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The First Integrals and Related Properties of a Class of Quintic Systems with a Uniform Isochronous Center*

Yuexin Yan[1] and Zhengxin Zhou[2,†]

Abstract In this paper, we give all the first integrals of the quintic systems which have a uniform isochronous center, and use them to determine the qualitative behavior of the periodic solutions of their equivalent non-autonomous systems. Meanwhile, we clearly describe the local phase portraits of singularity at infinity.

Keywords First integrals, Equivalence system, Uniform isochronous center, Infinite singularity, Local phase portraits.

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1. Introduction

Consider a planar polynomial differential system

$$\begin{cases} x' = ax + by + \sum_{k=2}^{\infty} P_k(x, y), \\ y' = cx + dy + \sum_{k=2}^{\infty} Q_k(x, y), \end{cases} \tag{1.1}$$

where $P_k(x, y)$ and $Q_k(x, y)$ are real homogeneous polynomials in x and y of degree k, which is an integer and greater than or equal to two.

Determining a singular point of a planar polynomial differential system to be a center is called Poincaré center-focus problem, which has been exhaustively studied in the last century, and it is closely related to the Hilbert 16th problem. Nevertheless, in spite of all efforts, there is no general method to solve this problem. Up to now, only for quadratic systems and some special systems the center conditions have been obtained [1, 2, 9, 12, 16, 18, 21, 22, 31]. Poincaré and Lyapunov [13] have provided such criterion: The origin point of (1.1) (with $a + d = 0$, $ad - bc > 0$) is a center, if and only if it possesses a nonconstant real analytic first integral in a neighbourhood of the origin (or there is a nonzero analytic integrating factor in a neighbourhood of the origin). However, in general, it is very difficult to find the integrating factor or the first integral. As increasing difficulties are encountered in the process of finding the central condition, some scholars have been trying to look

†the corresponding author.
Email address: yxyan@yzu.edu.cn (Y. Yan), zxzhou@yzu.edu.cn (Z. Zhou)
[1]Guangling College of Yangzhou University, Yangzhou, Jiangsu 225000, China
[2]College of Mathematical Science, Yangzhou University, Yangzhou, Jiangsu 225002, China
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for the conditions under which system (1.1) possesses a special center. The composition center has been discussed by Alwash and Lloyd [5,7] (see, for instance, [5,7]). Discussions of isochronous centres [4,6,8] have recently been provided by Algaba [3], Conti [14] and Villarini [25] as well as by Christopher and Devlin [11], these works refer to a number of articles on the topic, and have mentioned systems such as (1.1).

From [19], we get that if a planar differential polynomial system $x' = X(x,y), y' = Y(x,y)$ of degree n has a center at the origin of coordinates, then this center is an uniform isochronous, if and only if we carry out a linear change of variables and a scaling of time, it can be written into the form

$$\begin{cases} x' = -y + xP(x,y), \\ y' = x + yP(x,y), \end{cases} \tag{1.2}$$

where $P = \sum_{k=1}^{n-1} P_k(x,y)$, and $P_k(x,y)$ is a homogeneous polynomial in x and y of degree k.

System (1.2) is called rigid system [1,5]. For the rigid system (1.2) with $P = P_1 + P_m$ or $P = P_2 + P_{2m}$, m is an arbitrary positive integer. In [1] and [29,30], the authors used different methods to obtain the center conditions and point out that this center is also a composition center and uniform isochronous center.

In [8, 13, 19, 20], the authors have discussed the center conditions and phase portraits of system (1.2) with $P(x,y)$ as a quadratic or a cubic polynomial. In this paper, we shall be primarily concerned with the quintic system, which has the form

$$\begin{cases} x' = -y + x(P_2(x,y) + P_4(x,y)), \\ y' = x + y(P_2(x,y) + P_4(x,y)), \end{cases} \tag{1.3}$$

where $P_k = \sum_{i+j=k} p_{ij}x^iy^j$, $(k = 2,4)$, p_{ij} are real constants. First, we will give the first integrals of this system, when it has a center at origin point. Secondly, we shall establish some time-varying systems, which are equivalent (with the coinciding reflecting function [23]) to (1.3), and we use this autonomous system to determine the qualitative behavior of the periodic solutions of their equivalent non-autonomous systems. Finally, we will describe the local phase portraits of singularity at infinity.

2. The first integral

In this section, we will apply the method of Darboux [10, 15, 17] to sufficiently discover many algebraic integrals of system (1.3) and construct its first integrals .

Let $f(x,y) \in C[x,y]$, $f(x,y)$ not be identically zero. The algebraic curve $f(x,y) = 0$ is an invariant algebraic curve ($f(x,y)$ called algebraic integral) of the polynomial system

$$x' = X(x,y),\ y' = Y(x,y), \tag{2.1}$$

if for some polynomial $h(x,y) \in C[x,y]$, we have

$$f_x(x,y)X(x,y) + f_y(x,y)Y(x,y) = h(x,y)f(x,y).$$

The polynomial $h(x,y)$ is called the cofactor of the invariant algebraic curve $f(x,y) = 0$.