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# Triple Positive Solutions for a Class of Fractional Boundary Value Problem System*

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**Abstract** In this paper, the solvability for the following fractional boundary value problem system

$$
\begin{aligned}
&{}^{C}D_{0+}^{\sigma_1}v_1(t) = f_1(t, v_2(t), D_{0+}^{\mu_1}v_2(t)), \quad 0 < t < 1,\\
&{}^{C}D_{0+}^{\sigma_2}v_2(t) = f_2(t, v_1(t), D_{0+}^{\mu_2}v_1(t)), \quad 0 < t < 1,\\
&{v_1}'(0) = bv_1(0), \quad {v_1}''(0) = 0, \quad {}^{C}D_{0+}^{\theta_1}v_1(1) = a \cdot {}^{C}D_{0+}^{\theta_2}v_1(\eta),\\
&{v_2}'(0) = bv_2(0), \quad {v_2}''(0) = 0, \quad {}^{C}D_{0+}^{\theta_1}v_2(1) = a \cdot {}^{C}D_{0+}^{\theta_2}v_2(\eta),
\end{aligned}
$$

is studied, where $a > 0$, $-1 < b < 0$, $2 < \sigma_1, \sigma_2 \le 3$, $0 < \eta < 1$, $0 < \mu_1, \mu_2 \le 1$, $0 < \theta_2 \le \theta_1 \le 1$, $f_1$, $f_2$: $[0,1] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ are continuous, ${}^{C}D_{0+}^{\sigma_1}v_1(t)$, ${}^{C}D_{0+}^{\sigma_2}v_2(t)$ are the Caputo fractional derivatives, and $D_{0+}^{\mu_1}v_2(t)$, $D_{0+}^{\mu_2}v_1(t)$ are the Riemann-Liouville fractional derivatives. The fixed point theorem is used to prove that there are three positive solutions to problems.

**Keywords** Fractional derivative, Boundary value problems, Fixed point theorem, Positive solutions.

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## 1. Introduction

Fractional calculus has been widely applied in abnormal diffusion, control, fluid mechanics, image processing characteristics and so on. Therefore, fractional differential equations have captured our attention and gradually become an important model to solve many practical problems. Scholars have studied the fractional boundary value problem from the local problem to the nonlocal problem [2,7,9], from the non resonance problem to the resonance problem [2], from the finite interval problem to the infinite interval problem [1,5,14], from the single equation [3,11,13] to the system of equations [3,4,9] and so on. In particular, we note the studies as follows.

Qin and Jia [9] studied the single equation boundary value problems with Caputo derivative

$$
{}^{C}D_{0+}^{\alpha}u(t) = h(t)f(t, u(t)), \quad 0 < t < 1, \tag{1.1}
$$

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$$u'(0) = bu(0),\ u''(0) = 0, \quad {}^C D_{0+}^{\beta_1} u(1) = a \cdot {}^C D_{0+}^{\beta_2} u(\eta), \tag{1.2}$$

where $2 < \alpha < 3$, $0 < \eta < 1$. The existence and multiplicity results are determined by the use of the Krasnosel'skill fixed point theorem.

Su [10] studied the system of equation boundary value problem with Riemann-Liouville fractional derivatives

$$D^\alpha u(t) = f(t, v(t), D^\mu v(t)), \quad 0 < t < 1, \tag{1.3}$$
$$D^\beta v(t) = g(t, u(t), D^\nu u(t)), \quad 0 < t < 1, \tag{1.4}$$
$$u(0) = u(1) = v(0) = v(1) = 0, \tag{1.5}$$

where $f$, $g$: $[0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $\mu$, $\nu > 0$, $1 < \alpha, \beta < 2$, $\alpha - \nu \geq 1$, $\beta - \mu \geq 1$. With the application of the Schauder fixed-point theorem, some existence results are obtained.

Bai and Ge [3] studied the boundary value problems with containing derivatives on the nonlinearity

$$x''(t) + f(t, x(t), x'(t)) = 0, \quad 0 < t < 1, \tag{1.6}$$
$$x(0) = x(1) = 0, \tag{1.7}$$

where $f$: $[0, 1] \times [0, +\infty) \times \mathbb{R} \to [0, +\infty)$ is continuous. With fixed point index theory, Bai established a new fixed point theorem in a cone. Imposing some growth conditions on the nonlinearity, the above boundary value problems have three positive solutions. The interesting point is that the nonlinear term depends on the derivative of the unknown function.

Inspired by the above pieces of literature, this paper mainly studies the system of the functional boundary value problems

$${}^C D_{0+}^{\sigma_1} v_1(t) = f_1(t, v_2(t), D_{0+}^{\mu_1} v_2(t)), \quad 0 < t < 1, \tag{1.8}$$
$${}^C D_{0+}^{\sigma_2} v_2(t) = f_2(t, v_1(t), D_{0+}^{\mu_2} v_1(t)), \quad 0 < t < 1, \tag{1.9}$$
$${v_1}'(0) = bv_1(0), \quad {v_1}''(0) = 0, \quad {}^C D_{0+}^{\theta_1} v_1(1) = a \cdot {}^C D_{0+}^{\theta_2} v_1(\eta), \tag{1.10}$$
$${v_2}'(0) = bv_2(0), \quad {v_2}''(0) = 0, \quad {}^C D_{0+}^{\theta_1} v_2(1) = a \cdot {}^C D_{0+}^{\theta_2} v_2(\eta), \tag{1.11}$$

where $a > 0$, $-1 < b < 0$, $2 < \sigma_1, \sigma_2 \leq 3$, $0 < \eta < 1$, $0 < \mu_1, \mu_2 \leq 1$, $0 < \theta_2 \leq \theta_1 \leq 1$ and $a\eta^{1-\theta_2}\Gamma(2 - \theta_1) < \Gamma(2 - \theta_2)$. The functions $f_1$, $f_2$: $[0, 1] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ are continuous, ${}^C D_{0+}^{\sigma_1} v_1(t)$, ${}^C D_{0+}^{\sigma_2} v_2(t)$ are the Caputo fractional derivatives, and $D_{0+}^{\mu_1} v_2(t)$, $D_{0+}^{\mu_2} v_1(t)$ are the Riemann-Liouville fractional derivatives. We will prove that problems (1.8)-(1.11) have three positive solutions under some growth conditions by using the fixed point theorem given in [3].

# 2. Preliminaries and lemmas

## 2.1. Theorem and lemma

**Lemma 2.1** ( [9]). *Given $\psi_i \in L[0, 1]$ (i=1, 2), the unique solution of the problem*

$${}^C D_{0+}^{\sigma_i} v_i(t) = \psi_i(t), \quad 0 < t < 1, i = 1, 2,$$
$${v_i}'(0) = bv_i(0),\ {v_i}''(0) = 0,\ {}^C D_{0+}^{\theta_1} v_i(1) = a \cdot {}^C D_{0+}^{\theta_2} v_i(\eta)$$