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Autonomous Planar Systems of Riccati Type

Gary R. Nicklason[1,†]

Abstract The role of Riccati type systems in the plane along with the related linear, second order differential equation is examined. If x and y are the variables of the Riccati differential equation, then any integrable Riccati system has two independent invariant curves dependent upon these variables whose nature is easily determined from the solution of the linear equation. Each of these curves has the same cofactor. Other invariant curves depend upon x alone and are shown to be less important. The systems have both Liouvillian and non–Liouvillian solutions and are easily transformable to symmetric systems. However, systems derived from them may not be symmetric in their transformed variables. Several systems from the literature are discussed with regard to the forms of the invariant curves presented in the paper. The relation of certain Riccati type systems is considered with respect to Abel differential equations.

Keywords Riccati differential equation, Centre–focus problem, Algebraic invariant curve, Cofactor, Symmetric centres.

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1. Introduction

In this work, we consider differential polynomial systems in the plane having the form

$$\begin{aligned}\frac{dx}{dt} &= -N(x,y),\\ \frac{dy}{dt} &= M(x,y)\end{aligned} \tag{1.1}$$

for polynomials M, N and specifically their relation to Riccati systems for which $M(x,y) = P(x)y^2 + Q(x)y + R(x)$, $N(x,y) = N(x)$. Our primary interest will be for the centre–focus cases

$$\begin{aligned}M(x,y) &= x + q(x,y),\\ N(x,y) &= y + p(x,y),\end{aligned} \tag{1.2}$$

where p, q are homogeneous polynomials of degree $n \geq 2$ or

$$\begin{aligned}M(x,y) &= x + q_2(x,y) + q_3(x,y),\\ N(x,y) &= y + p_2(x,y) + p_3(x,y),\end{aligned} \tag{1.3}$$

† the corresponding author.
Email address: gary_nicklason@cbu.ca (G. R. Nicklason)
[1] Department of Mathematics, Physics and Geology, Cape Breton University, Sydney, Nova Scotia, B1P 6L2, Canada

where p_2, q_2 and p_3, q_3 are homogeneous polynomials of degree 2 and 3 respectively and how these systems relate to the Riccati systems. We shall refer to the first of these as homogeneous systems and to the second as cubic systems. We give examples of this type that can be reduced to Riccati type systems. Associated with (1.1) is the ordinary differential equation

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}. \tag{1.4}$$

In these, we assume the variables along with any associated parameters in the differential equations are real, although some of the latter could be complex.

The cubic system (1.3) is a particular case of a more general system of centre–focus type in which the nonlinearity is expressed, as the sum of homogeneous polynomials having degrees n and $2n-1$ for integers $n \geq 2$. The cubic system corresponds to $n = 2$. In [22], the authors use a relation to an Abel differential equation to consider certain centre conditions for the quintic $n = 3$ system.

A point (x_0, y_0) is said to be a critical point of (1.1), if $M(x_0, y_0) = N(x_0, y_0) = 0$. This point is said to be a centre if all trajectories of the system on a neighbourhood of the critical point are closed. In his original work [20], Poincaré developed a method for determining, if the origin is a centre by seeking an analytic solution to (1.4), where M, N satisfy $M(0,0) = N(0,0) = 0$. This is given by

$$U(x,y) = \frac{1}{2}(x^2 + y^2) + \sum_{k=3}^{\infty} U_k(x,y), \tag{1.5}$$

where the U_k are homogeneous polynomials of degree k. The solution (1.5) is required to satisfy the condition

$$\frac{dU}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} = \sum_{k=2}^{\infty} V_k (x^2 + y^2)^k.$$

The V_k are called *Lyapunov coefficients* and they are homogeneous polynomials in the coefficients of the system. A necessary and sufficient condition for the critical point $(0,0)$ to be a centre is the vanishing of all the Lyapunov coefficients. One way of finding centre conditions is to compute several of the V_k, and then obtain necessary conditions for them to vanish. From this, one hopes to show that these conditions are sufficient so that all $V_k = 0$. In this case, (1.5) will have a form which is convergent and the solution will be given by $U(x,y) = C$ where C is a constant. Another approach is to show that (1.4) can be solved. The main method for doing this is the *Darboux method*. This approach has been well studied and a great number of general results concerning it are known. It requires the existence of *algebraic invariant curves* which are used to construct integrating factors, and in some cases, solutions. An integrating factor of (1.4) is a function $\mu(x,y)$ which satisfies the partial differential equation

$$-N(x,y)\frac{\partial \mu}{\partial x} + M(x,y)\frac{\partial \mu}{\partial y} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mu. \tag{1.6}$$

An *algebraic invariant curve* of a system (1.1) is an expression of the form $f(x,y) = 0$ where f is a polynomial. It is required to satisfy the partial differential equation

$$-N(x,y)\frac{\partial f}{\partial x} + M(x,y)\frac{\partial f}{\partial y} = \lambda(x,y)f. \tag{1.7}$$