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The Monotonicity of the Linear Complementarity Problem*

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Abstract The monotonicity of the linear complementarity problem (LCP) is discussed in this paper. Both the monotone property about the single element of the solution and the monotone property of the whole solution are presented. In order to illustrate the results, some corresponding numerical experiments are provided.

Keywords Linear complementarity problem, Solution, Monotonicity.

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1. Introduction

The linear complementarity problem is to find a vector $z \in R^n$ such that

$$z^{\mathrm{T}}(Mz+q)=0, \quad z \geq 0, \quad Mz+q \geq 0, \tag{1.1}$$

where $M = (m_{ij}) \in R^{n\times n}$ and $q \in R^n$. Problem (1) is usually denoted by LCP(M, q), which has many applications such as the elastic contact problems, the free boundary problems, the linear and quadratic programming problems and the market equilibrium problems (see [1,3,7,10,15,21,26,27] and the references therein).

The theory research and the numerical algorithms for the LCP(M, q) have been studied in recent decades. The theory research includes the existence and uniqueness of the solution, the stability and sensitivity of the solution, the relationship between the LCP(M, q) and other problems, etc., (see [6,7,10,14,18,20,21,23,24]). It is well-known that the LCP(M, q) has a unique solution for any $q \in R^n$, if and only if the system matrix M is a P-matrix. The positive definite matrix and the H_+-matrix are two types of P-matrices, both of which have been studied by many authors (see [1, 9, 10, 14, 19]). For the stability and sensitivity of the solution, Mathias, Pang, Cottle and other researchers discussed the error problem and the perturbation problem of the solution, and many interesting results have been obtained, including the Lipschitzian continuous property of the solution (see [4,5,7,8,12,16,17,23,25]). For the numerical algorithms of the LCP(M, q), all kinds of solving methods have been presented, including the direct methods and the iteration methods such as

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the Lemke method, the projected method, and the modulus-based matrix splitting iteration method, etc.. Most of the solving methods are very efficient, and for the detailed materials, readers can refer to [1, 13, 15, 27] and the references therein.

Although there are many theories for the LCP(M, q), there are few theoretical studies on the monotony. In this paper, we will study the monotonicity problem of the solution. We will present that the solution possesses the monotone decreasing property for an arbitrary single variable, when matrix M is a P-matrix and the whole solution is monotone decreasing under some conditions when the system matrix M is an M-matrix. Besides, some conclusions related to the solution and the corresponding experiments will be provided.

The outline of this paper is as follows. We briefly introduce some definitions, then present the main conclusions in Section 2. The numerical experiments are shown and discussed in Section 3. We end this paper by some concluding remarks in Section 4.

2. Preliminaries and main results

First, we review several definitions as follows.

Definition 2.1. (Murty [19], Gale and Nikaido [11]) M is said to be a P-matrix, if all its principal minors are positive.

Definition 2.2. (Ostrowski [22], Berman and Plemmons [2]) A matrix $M \in R^{n\times n}$ is called an M-matrix, if

$$M^{-1} \geq 0 \quad \text{and} \quad m_{ij} \leq 0 (i \neq j) \quad for \quad i, j = 1, 2, ..., n. \tag{2.1}$$

Definition 2.3. (Murty [20]) Let $M \in R^{n\times n}$ be a matrix, the complementarity set of column vectors of M is a set $\{A_{.j}, j = 1, 2, ..., n\}$ such that $A_{.j}$ is either $I_{.j}$ or $-M_{.j}$, for each $j = 1, 2, ..., n$.

Definition 2.4. (Murty [20]) Let $M \in R^{n\times n}$ be a matrix and $\{A_{.j}, j = 1, 2, ..., n\}$ be any complementarity set of column vectors of M, a complementarity cone of M is the set

$$pos\{A_{.j}, j = 1, 2, ..., n\} = \{\sum_{j=1}^{n} \beta_j A_{.j}, \beta_j \geq 0, j = 1, 2, ..., n\}. \tag{2.2}$$

In the following, we give the main conclusions of this paper.

Theorem 2.1. *Suppose $M \in R^{n\times n}$ is a P-matrix, and $\hat{q}, \tilde{q} \in R^n$ satisfy $\hat{q}, \tilde{q} \in pos\{A_{.j}, j = 1, 2, ..., n\}$, which is a complementarity cone of M. If the solutions of (1) with $q = \hat{q}, \tilde{q}$ are denoted by $\hat{z}, \tilde{z}$ respectively, then*

(i) $\lambda\hat{q} + \mu\tilde{q} \in pos\{A_{.j}, j = 1, 2, ..., n\}, \lambda \geq 0, \mu \geq 0$;

(ii) $\lambda\hat{z} + \mu\tilde{z}$ *is the solution of (1), when* $q = \lambda\hat{q} + \mu\tilde{q}$.

Proof. The conclusion (i) can be easily proved based on the definition of the complementarity cone of M. We only prove (ii) in the following.