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Existence and Multiplicity of Positive Solutions for a Singular Nonlinear High Order Fractional Differential Problem with Multi-point Boundary Conditions*

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Abstract In this paper, a singular nonlinear high order fractional differential problem involving multi-point boundary conditions is solved by means of the fixed point index theory. Some properties of the first eigenvalue corresponding to relevant operator and some new height functions are also used to prove the existence and multiplicity of positive solutions. The nonlinearity depends on arbitrary fractional derivative.

Keywords Positive solutions, Fractional differential problem, Fixed point index, First eigenvalue, Multi-point boundary conditions.

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1. Introduction

We strive to investigate the existence and multiplicity of positive solutions for the following fractional equation problem:

$$\begin{cases} D_{0+}^{\alpha}y(t)+g(t)f(t,y(t),D_{0+}^{w}y(t))=0, \quad 0<t<1,\\ D_{0+}^{w}y(0)=D_{0+}^{w+1}y(0)=\cdots=D_{0+}^{w+n-2}y(0)=0,\\ D_{0+}^{\beta}y(1)=\displaystyle\sum_{i=1}^{m}\eta_i D_{0+}^{\gamma}y(\zeta_i), \end{cases} \tag{1.1}$$

where $\alpha\in\mathbb{R}$, $n-1<\alpha\le n$, $n>2$, $\eta_i\ge 0$, $i=1,2,\cdots,m$ ($m\in\mathbb{N}^+$), $0<\zeta_1<\zeta_2<\cdots<\zeta_m<1$, $\beta,\gamma\in\mathbb{R}$, $1\le\beta-w$, $\beta\le n-2$ and $0\le\gamma\le\beta$ with $(\Gamma(\alpha)/\Gamma(\alpha-\gamma))\sum_{i=1}^{m}\eta_i\zeta_i^{\alpha-\gamma-1}<\Gamma(\alpha)/\Gamma(\alpha-\beta)$, $0\le w\le 1$, D_{0+}^{α} is the α-order Riemann-Liouville derivative, $f(t,u,v)\in C([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty))$, $g(t)$ is continuous and may have singularities at the points $t=0,1$. Under certain conditions, by using some properties of the first eigenvalue corresponding to relevant operator, the different height functions of the nonlinear term of the equation defined on the special bounded set and theory of the fixed point index, we obtain the existence and multiplicity results of positive solutions.

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In recent years, a large number of fractional differential equations with various boundary conditions have been paid attention to people in many fields such as science and engineering. This is mainly because in most cases we can use such a mathematical model to accurately and quickly solve many complex problems in various fields, such as biology physics, chemistry, control theory, engineering, mechanics, aerodynamics and other fields. For details, see [3, 4, 13, 21, 23, 29]. Recently, extensive research on differential equations has promoted the development of boundary value problems (BVP) of differential equations. They include singular BVP [5, 10, 17, 19, 22, 30], semipositone BVP [1, 9, 16, 24–26] and nonlocal BVP [2, 8, 11, 12, 15, 18, 27] as special cases. The existence, uniqueness and multiplicity of solutions to these problems are obtained by using nonlinear analysis techniques such as the nonlinear alternative technique, fixed point theorems, the method of monotone iterative, upper and lower solutions method. Now, we give some examples. In [19], Jiang et al. explored the following two-term fractional equation problem with two point boundary value problem:

$$\begin{cases} -D_{0+}^{\alpha}x(t) + ax(t) = b(t)f(t, x(t)), \quad 0 < t < 1, \\ x(0) = 0, \quad x(1) = 0, \end{cases}$$

where $1 < \alpha \leq 2$, $a > 0$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, $b(t)$ is continuous and its singularities are $t = 0, 1$, by virtue of u_0-positive operator and theory of the fixed point index, at least one positive solution has been found. In [28], Zhang *et al.* studied the following differential equation, which is an integral boundary value problem:

$$\begin{cases} D_{0+}^{\alpha}x(t) + f(t, x(t)) = 0, \quad 0 < t < 1, \\ x^{(\beta)}(0) = 0, \quad 0 \leq \beta \leq n-2, \\ [D_{0+}^{\gamma}x(t)]_{t=1} = \lambda \displaystyle\int_0^{\eta} g(t) D_{0+}^{\gamma}x(t)dt, \end{cases}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative, $n-1 < \alpha \leq n$, $n \geq 3$, $\gamma \geq 1$, $\alpha - \gamma - 1 > 0$, $\eta \in (0,1]$, $0 \leq \lambda \int_0^{\eta} g(t)t^{\alpha-\gamma-1}dt < 1$, $g \in L^1[0,1]$ is nonnegative, the singularities of $f(t,x)$ are $t = 0, 1$ and $x = 0$. By using Leggett-Williams fixed point theory, the authors demonstrated that the equation at least has three positive solutions. In [12], He et al. explored the following differential problem with the Riemann-Stieltjes integral and with any derivative in the integral:

$$\begin{cases} D_{0+}^{\alpha}x(t) + f(t, x(t), D_{0+}^{\beta}x(t)) = 0, \quad 0 < t < 1, \\ D_{0+}^{\beta}x(0) = D_{0+}^{\beta+1}x(0) = 0, \\ [D_{0+}^{\gamma}x(t)]_{t=1} = \displaystyle\int_0^1 D_{0+}^{\gamma}x(s)dA(s), \end{cases}$$

where D_{0+}^{α} is the Riemann-Liouville derivative, $2 < \alpha \leq 3$, $0 < \beta \leq \gamma < \alpha - 2$, $\int_0^1 D_{0+}^{\gamma}x(s)dA(s)$ is a linear functional with the Riemann-Liouville integrals, the singularities of $f(t,u,v)$ are $t = 0, 1$ and $u = v = 0$, by applying suitable upper and lower solutions and Schauder's fixed point theorem, the authors proved that the problem at least has one positive solution.

Motivated by all the papers above, we discuss the existence and multiplicity of positive solutions of (1.1). Our article have various features. Firstly, the equation