Journal of Nonlinear Modeling and Analysis
Volume 3, Number 3, September 2021, 403–419
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2021.403

Hopf Bifurcation Analysis for a Delayed Business Cycle Model-The Equivalence of Multiple Time Scales Versus Center Manifold Reduction Methods*

Xiaolin Zhen[1] and Yuting Ding[1,†]

Abstract In this paper, we study the Hopf bifurcation of a model with a second order term, which is the business cycle model with delay. Multiple time scales method, which is mainly used by the engineering researchers, and center manifold reduction method, which is mainly used by researchers from mathematical society, are used to derive the two types of normal forms near the Hopf critical point. A comparison between the two methods shows that the two normal forms are equivalent. Scholars can derive the normal form by choosing appropriate methods according to their actual demands. Moreover, bifurcation analysis and numerical simulations are given to verify the analytical predictions.

Keywords Business cycle, Hopf bifurcation, Normal form, Multiple time scales, Center manifold reduction.

MSC(2010) 34K18, 37L10.

1. Introduction

Over the past several decades, the bifurcation problems of delayed differential equations have been hot topics in studying nonlinear dynamical systems. As we all know, it is very important to compute normal forms of differential equations in study of bifurcation properties. There are two popular and effective approaches for determining the normal forms of bifurcations in nonlinear delayed dynamical systems, that is, multiple time scales (MTS) method [11, 12] and center manifold reduction (CMR) method [5, 6, 16].

The MTS method was originally used to study the Hopf bifurcation of one-dimensional second order ordinary differential vibration equation [11], and Nayfeh [12] extended this method to solve Hopf bifurcation of functional differential equations. The MTS method is mainly used by applied scientists and researchers from engineering society since it is simple without complicated computation [1,2,11–13,15], while this method has some limitations. For example, it cannot solve

†the corresponding author.
Email address: xiannvzl@163.com (X. Zhen), yuting840810@163.com (Y. Ding)
[1]Department of Mathematics, Northeast Forestry University, Harbin, Heilongjiang 150040, China
*The authors were supported by Fundamental Research Funds for the Central Universities (Grant No. 2572019BC14) and the Heilongjiang Provincial Natural Science Foundation (Grant No. LH2019A001).

non-semisimple singularity bifurcation in functional differential equations, such as Bogdanov-Takens bifurcation.

The CMR method can be used to solve all kinds of functional differential equations [7, 9, 17], and it is widely used by researchers from mathematical society. However, this method needs the basic knowledge of functional analysis and algebra in mathematics major, and it needs large amount of calculation and complicated process. For CMR method in delayed differential systems, one needs to first change the delayed equations to an abstract ordinary differential equation in infinite dimensional image space, and then decompose the solution space of their linearized form into stable manifold and center manifold. Next, with adjoint operator equations, one computes the center manifold by projecting the whole space to the center manifold,and finally calculates the normal form restricted to the center manifold.

In fact, both of the two approaches combine the two steps involved in using center manifold theory and normal form theory into one unified step to obtain the normal form and nonlinear transformation simultaneously, thus, there may exist some relations between the two methods. Many authors considered some types of bifurcations by using the two methods at the same time. For example, Nayfeh [12] used both the MTS and CMR methods to derive equivalent normal forms of Hopf bifurcation for some simple delayed nonlinear dynamical systems, while the CMR method used in this paper had some differences with Faria's center manifold reduction method. Ding et al. [3, 4] applied the two methods to obtain the normal forms near Hopf-zero and double-Hopf critical points in delayed differential equations respectively, and showed their equivalence. Peng et al. [14] used two methods to study the Hopf bifurcation of van der Pol-Duffing equation with delay, while in this paper, Peng et al. only used Hassard's method [8] to derive the formulae for determining the stability of Hopf bifurcating periodic solutions and the direction of Hopf bifurcation, not showed the explicit normal form of Hopf bifurcation. Moreover, we find that the systems discussed in the above papers are all without quadratic terms. Actually, Yu et al. [18] proved that, if system does not contain second-order terms, the normal forms associated with the semisimple n_1-Hopf-n_2-zero singularity, derived by using the multiple time scales and center manifold reduction methods, are identical up to third order.

However, if system contains second-order terms, can we also obtain equivalent normal forms by using MTS and CMR methods? It is also the motivation of this study. In this paper, we consider the following delayed business cycle model, which contains quadratic term [10]. Then, we investigate the equivalence of two normal forms of Hopf bifurcation in this system, derived by using the multiple time scales and center manifold reduction methods, and the system shows as follows:

$$\ddot{x} + ax(t-\tau) - qx^3 = -v\dot{x}^3 - v\dot{x}^2 - u\dot{x}, \tag{1.1}$$

where x represents gross national income(or output), $\dot{x}$ is the derivative of x to time t; $0 < a < 1$ is marginal propensity to consume; $q > 0$ is fixed interest rate; $v > 0$ denotes the fixed rate of national income, we usually call it acceleration factor; $0 < u \leq 1$, $1/u$ is called Keynesian coefficient due to the time lag of investment process, τ represents the delay caused by the delay of investment decision. In the business cycle model, gross national income is an important target to reflect the overall economic activities, which is often used in the research of macroeconomic. Thus, the delay is introduced into gross national income, and the results show that the delay will cause the fluctuation of macroeconomics and effect the stability of