Ann. of Appl. Math.
33:4(2017), 391-399

THE SEMI-NORMS ON THE SCHWARTZ SPACE*†

Dan Mu‡, Changmao Li

(*School of Science, Hubei University for Nationalities, Enshi 445000, Hubei, PR China*)

Abstract

Let $S(R^2)$ be the class of all infinitely differential functions which, as well as their derivatives, are rapidly decreasing on R^2. Here we define a kind of semi-norms which is equivalent to the usual family of semi-norms on the Schwartz space $S(R^2)$.

Keywords Schwartz space; semi-norms; equivalent

2000 Mathematics Subject Classification 46A11

1 Introduction

In the recent years, the Schwartz space as well as their application are concerned in many publication ([1-5]). In this paper, we first give the usual family of semi-norms on the Schwartz space $S(R^2)$. A new family of semi-norms is defined, which is based on the operators we constructed.

Using the new family of semi-norms, we can consider the method to discuss the Schwartz space in terms of the sequential theory.

Let I_+^2 denote the set of all two-tuple of non-negative integers. For $\alpha \in I_+^2$, let

$$|\alpha| = \alpha_1 + \alpha_2. \tag{1.1}$$

For a multi-index α and $x \in R^2$, let

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}, \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}. \tag{1.2}$$

The Schwartz space $S(R^2)$ is defined to be the class of all infinitely differentiable complex-valued functions φ on R^2 such that

$$\lim_{|x|\to\infty} |x^\alpha D^\beta \varphi| = 0, \tag{1.3}$$

for all multi-indices α and β. The space $S(R^2)$ is closed for the differential operators and multiplication by polynomials.

---

*The work was supported by National Natural Science Foundation (NNSF) of China (11761030), Hubei Provincial Department of Education Science and Technology Research Project (B2015099) and the Doctoral Science Foundation of Hubei University for Nationalities (4148009).

†Manuscript received June 27, 2016; Revised July 25, 2017

‡Corresponding author. E-mail: bai_mudan@hotmail.com

2 Some Definitions

In this section, we introduce some definitions. Let R^2 be the 2-dimensional Euclidean space.

Definition 2.1 A semi-norm on a vector space V is a map $\rho : V \to [0,\infty)$ satisfying

(i) $\rho(u+v) \le \rho(u)+\rho(v)$ for $u,v \in V$;

(ii) $\rho(au) = |a|\rho(u)$ for $a \in C$(or R).

A family of semi-norms $\{\rho_\alpha\}_{\alpha\in A}$ is said to be separate points if

(iii) $\rho_\alpha(u) = 0$ for all $\alpha \in A$ implies $u = 0$,

where $\alpha = (\alpha_1, \alpha_2)$ are two-tuple of non-negative integers ([1]).

Definition 2.2 Let $f \in S(R^2)$ and $\|\cdot\|_{\alpha,\beta,\infty}$ be defined by

$$\begin{aligned}\|f\|_{\alpha,\beta,\infty} = \|x^\alpha D^\beta f\|_\infty &= \Big\| x_1^{\alpha_1} x_2^{\alpha_2} \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1}\partial x_2^{\beta_2}} f \Big\|_\infty \\ &= \sup_{x_1\in R}\sup_{x_2\in R}\Big| x_1^{\alpha_1} x_2^{\alpha_2} \frac{\partial^{(\beta_1+\beta_2)}}{\partial x_1^{\beta_1}\partial x_2^{\beta_2}} f \Big|. \end{aligned} \tag{2.1}$$

Then $\{\|\cdot\|_{\alpha,\beta,\infty}\}_{\alpha,\beta\in I_+^2}$ is the usual family of semi-norms on $S(R^2)$.

Definition 2.3 Let V be a vector space and $u \in V$. Let $\{\rho_\alpha\}_{\alpha\in A}$ and $\{d_b\}_{b\in B}$ be two families of semi-norms on a vector space V where A and B are some index sets. The families of semi-norms are equivalent if and only if they satisfy:

(i) For each $a \in A$, there exist $b_1, b_2 \in B$ and $C > 0$, such that

$$\rho_a(u) \le C\big(d_{b_1}(u) + d_{b_2}(u)\big);$$

(ii) for each $b \in B$, there exist $a_1, a_2 \in A$ and $C' > 0$ such that

$$d_b(u) \le C'\big(\rho_{a_1}(u) + \rho_{a_2}(u)\big).$$

Definition 2.4 A family of semi-norms $\{\rho_\alpha\}_{\alpha\in A}$ on a vector space V is called directed if for $\alpha, \beta \in A$ and $u \in V$, there exist $\gamma \in A$ and $C > 0$ such that

$$\rho_\alpha(u) + \rho_\beta(u) \le C\rho_\gamma(u). \tag{2.2}$$

Definition 2.5 Let $f \in S(R^2)$ and $\|\cdot\|_{\alpha,\beta,2}$ be define by

$$\|f\|_{\alpha,\beta,2} = \|x^\alpha D^\beta f\|_2 = \Big(\int_{R^2} |x^\alpha D^\beta f(x)|^2 \mathrm{d}x\Big)^{\frac{1}{2}}.$$

Then $\{\|\cdot\|_{\alpha,\beta,2}\}_{\alpha,\beta\in I_+^2}$ is the usual family of semi-norms on $S(R^2)$.

Definition 2.6 Hölder inequality: Let E be a measurable set of Lebesgue, $x(t)$ and $y(t)$ be measurable functions in E. Then p and q are positive numbers such that $\frac{1}{p}+\frac{1}{q} = 1$, then