Ann. of Appl. Math.
33:3(2017), 324-330

NORM RETRIEVAL BY PROJECTIONS ON INFINITE-DIMENSIONAL HILBERT SPACES*†

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Abstract

We study the norm retrieval by projections on an infinite-dimensional Hilbert space H. Let $\{e_i\}_{i\in I}$ be an orthonormal basis in H and $W_i = \{e_i\}^{\perp}$ for all $i \in I$. We show that $\{W_i\}_{i\in I}$ does norm retrieval if and only if I is an infinite subset of $\mathbf{N}$. We also give some properties of norm retrieval by projections.

Keywords norm retrieval; phase retrieval; frames; Hilbert spaces
2000 Mathematics Subject Classification 42C15

1 Introduction

Signal reconstruction is an important problem in engineering and has a wide variety of applications. Recovering a signal when there is partial loss of information is a significant challenge. Partial loss of phase information occurs in application areas such as speech recognition [4, 12, 13], and optics applications such as X-ray crystallography [3, 10, 11], and there is a need to do phase retrieval efficiently. The concept of phase retrieval for Hilbert space frames was introduced in 2006 by Balan, Casazza, and Edidin [2], and since then it has become an active area of research in signal processing and harmonic analysis.

Phase retrieval has been defined for vectors as well as for projections and in general deals with recovering the phase of a signal given its intensity measurements from a redundant linear system. Phase retrieval by projections, where the signal is projected onto some higher dimensional subspaces and has to be recovered from the norms of the projections of the vectors onto the subspaces, appears in real life problems such as crystal twinning [9]. We refer the readers to [5] for the detailed study of phase retrieval by projections.

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*The work was supported in part by the Foundation of Fuzhou University (Grant No.JA15059).
†Manuscript received May 27, 2017; Revised July 4, 2017
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In this paper, we consider the notion of norm retrieval which was recently introduced by Bahmanpour et. al. [1], and is the problem of retrieving the norm of a vector given the absolute value of its intensity measurements. Norm retrieval arises naturally from phase retrieval when one utilizes both a collection of subspaces and their orthogonal complements. Let $\{e_i\}_{i\in I}$ be an orthonormal basis in Hilbert space H. Let $W_i = \{e_i\}^{\perp}$ for all $i \in I$. [7] discussed that $\{W_i\}_{i\in I}$ does norm retrieval or cannot do norm retrieval in a finite dimensional Hilbert space H. [7] also gave some properties of norm retrieval. We will discuss the same problems but in an infinite-dimensional Hilbert space H. We will also give some results similar to these in [7].

2 Norm Retrieval

Firstly, we give some definitions.

Definition 2.1 A family of vectors $\{f_i\}_{i\in I}$ in the infinite-dimensional Hilbert space H is a frame if there are constants $0 < A \le B < +\infty$ so that for all $x \in H$,

$$A\|x\|^2 \le \sum_{i\in I}|\langle x, f_i\rangle|^2 \le B\|x\|^2.$$

If $A = B$, the frame is called an A-tight frame (or a tight frame).

Note that the theory of frames in Hilbert spaces presents a central tool in mathematics and engineering, and has been developed rather rapidly in the past decade [6, 8].

Definition 2.2 Let $\{W_i\}_{i\in I}$ be a collection of subspaces in the infinite-dimensional Hilbert space H and $\{P_i\}_{i\in I}$ be the orthogonal projections onto each of these subspaces. We say that $\{W_i\}_{i\in I}$ (or $\{P_i\}_{i\in I}$) yields phase retrieval if for all $x, y \in H$ and $i \in I$, $\|P_ix\| = \|P_iy\|$, then $x = cy$ for some scalar c with $|c| = 1$.

In particular, a set of vectors $\{f_i\}_{i\in I}$ in H does phase retrieval, if for $x, y \in H$ and $i \in I$, $|\langle x, f_i\rangle| = |\langle y, f_i\rangle|$, then $x = cy$ for some scalar c with $|c| = 1$.

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Remark 2.1 It is immediate that a collection of subspaces (or orthogonal projections) or a family of vectors doing phase retrieval does norm retrieval.

It is easy to see that tight frames do norm retrieval.

Theorem 2.1 *Tight frames do norm retrieval.*

Proof Let $\{f_i\}_{i=1}^{\infty}$ in the infinite-dimensional Hilbert space H be an A-tight