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ON THE NORMALIZED LAPLACIAN SPECTRUM OF A NEW JOIN OF TWO GRAPHS*

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Abstract

Given graphs G_1 and G_2, we define a graph operation on G_1 and G_2, namely the SSG-vertex join of G_1 and G_2, denoted by $G_1 \star G_2$. Let $S(G)$ be the subdivision graph of G. The SSG-vertex join $G_1 \star G_2$ is the graph obtained from $S(G_1)$ and $S(G_2)$ by joining each vertex of G_1 with each vertex of G_2. In this paper, when G_i $(i = 1, 2)$ is a regular graph, we determine the normalized Laplacian spectrum of $G_1 \star G_2$. As applications, we construct many pairs of normalized Laplacian cospectral graphs, the normalized Laplacian energy, and the degree Kirchhoff index of $G_1 \star G_2$.

Keywords spectrum; SSG-vertex join; normalized Laplacian cospectral graphs; normalized Laplacian energy; degree Kirchhoff index

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1 Introduction

All graphs described in this paper are simple and undirected. Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \cdots, e_m\}$. The adjacency matrix of G is $A(G) = (a_{ij})_{n\times n}$ with $a_{ij} = 1$ if v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise. Let $D(G) = \mathrm{diag}(d_1, d_2, \cdots, d_n)$ be the diagonal matrix of vertex degrees of G. The Laplacian matrix and the normalized Laplacian matrix are defined as $L(G) = D(G) - A(G)$ and $\mathscr{L}(G) = D(G)^{-\frac{1}{2}}L(G)D(G)^{-\frac{1}{2}}$, respectively. The characteristic polynomial of $\mathscr{L}(G)$ is defined as $f_G(\mathscr{L}(G) : x) = \det(xE_n - \mathscr{L}(G))$, where E_n is the identity matrix of order n. The roots of $f_G(\mathscr{L}(G) : x) = 0$ is called the normalized Laplacian eigenvalues of G, denoted by $\gamma_1(G), \gamma_2(G), \cdots, \gamma_n(G)$, where $\gamma_n(G) = 0$. Since $A(G)$ is a real symmetric matrix, its eigenvalues are all real numbers. Their eigenvalues are conventionally denoted and arranged as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. The set of

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all the eigenvalues together with their multiplicities is called the A-spectrum of G. Graphs G_1 and G_2 are called normalized Laplacian cospectral if they have the same normalized Laplacian spectrum. The incidence matrix R of G is a $(0,1)$ matrix with rows indexed by vertices and column indexed by edges, where $R_{ve} = 1$ when the vertex v is an end point of the edge e and 0 otherwise. The subdivision graph of G, denoted by $S(G)$, is the graph obtained by inserting a new vertex into every edge of G. Undefined terminology and notations may refer to [1].

Spectral graph theory is a fast growing branch of algebraic graph theory and concerns an interwind tale of properties of graphs and spectrum of related matrices. Calculating the spectra of graphs as well as formulating the characteristic polynomials of graphs is a fundamental and very meaningful work in spectra graph theory. The characteristic polynomial and spectra of graphs help to investigate some properties of graphs such as energy [2,3], the Kirchoff index [4-6], the Laplacian-energy-like invarients [7,8], the normalized energy [15,16], the degree Kirchoff index [17], and so on. Recently, many graph operations such as the subdivision join, the corona, the edge corona and the neighbour corona have been introduced, and their spectrum are computed [9-13].

Motivated by the above works, given graphs G_1 and G_2, we define a new join of graphs $G_1 \star G_2$ and obtain their normalized Laplacian spectrum when G_1 and G_2 are regular graphs. As applications, we construct many pairs of normalized Laplacian cospectral graphs, the normalized Laplacian energy, and the degree Kirchhoff index of $G_1 \star G_2$.

2 Normalized Laplacian Spectrum of A New Join of Two Graphs

In this section, we first define a new join of graphs $G_1 \star G_2$ and list some known results, then determine the normalized Laplacian spectrum of $G_1 \star G_2$ when G_1, G_2 are regular graphs. Moreover, we also construct many pairs of normalized Laplacian cospectral graphs, the normalized Laplacian energy, and the degree Kirchhoff index of $G_1 \star G_2$. Let $\mathbf{1}_k$ and $\mathbf{0}_k$ be two vectors of order k with all elements equal to 1 and 0, respectively. Moreover, $\mathbf{J}_{m\times n}$ denotes all $m \times n$ ones matrix, and $\mathbf{0}_{m\times n}$ denotes all $m \times n$ zeros matrix.

Definition 2.1 The SSG-vertex join of graphs G_1 and G_2, denoted by $G_1 \star G_2$, is the graph obtained from the subdivision graphs $S(G_1)$ and $S(G_2)$ by joining each vertex of $V(G_1)$ with each vertex of $V(G_2)$.

Note that if we consider the graphs G_i with n_i vertices and m_i edges $(i = 1, 2)$, then the graph $G_1 \star G_2$ in Definition 2.1 contains $n_1 + m_1 + n_2 + m_2$ vertices and $2m_1 + 2m_2 + n_1n_2$ edges. For example, the graphs $P_2 \star K_3$ is depicted in Figure 1.