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SOME NEW DISCRETE INEQUALITIES OF OPIAL WITH TWO SEQUENCES*†

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Abstract

In this paper, we establish some new discrete inequalities of Opial-type with two sequences by making use of some classical inequalities. These results contain as special cases improvements of results given in the literature, and these improvements are new even in the important discrete case.

Keywords Opial type inequality; Hölder's inequality; forward difference operator; backward difference operator

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1 Introduction

In 1960, Opial [1] established the following important integral inequality:

Theorem A *Let $f(x) \in C^1[0, h]$ be such that $f(0) = f(h) = 0$, and $f(x) > 0$ in $(0, h)$. Then*

$$\int_0^h |f(x)f'(x)|\mathrm{d}x \le \frac{h}{4}\int_0^h (f'(x))^2\mathrm{d}x, \tag{1.1}$$

where $\frac{h}{4}$ is the best possible.

Inequality (1.1) is known in the literature as Opial inequality. For the past few years, Opial's inequality and its generalizations, extensions and discretizations have played a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations. Many mathematicians gave the improvements and generalizations in last few decades to add the considerable contribution in the literature (see, for instance, [2-10]).

The paper is motivated by the work of K.C. Hsu and K.L. Tseng [11] and their discrete versions of Opial type inequalities about forward difference operator. We

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will prove some new discrete inequalities of Opial type with two sequences. In this paper, we denote $\{x_i\}_{i=0}^l$ by a sequence of real numbers, the operators Δ and ∇ by $\Delta x_i = x_{i+1} - x_i$ and $\nabla x_i = x_i - x_{i-1}$ respectively. In the next section, we will give the improvements of discrete inequalities of Opial type.

2 Main Results

Throughout this section, let $m,\ n > 0$ and $c(m, n) = \frac{1}{m+n} \max\{m,\ n\}$. We start with the following theorem.

Theorem 2.1 *Let $\{x_i\}_{i=0}^l$ and $\{y_i\}_{i=0}^l$ be sequences of real numbers with $x_0 = y_0 = 0$. Then, the following inequality holds*

$$\sum_{i=1}^{l} \big(|x_i|^n |\Delta y_i|^m + |y_i|^n |\Delta x_i|^m\big) \le c(m, n) l^n \sum_{i=0}^{l} \big(|\Delta x_i|^{m+n} + |\Delta y_i|^{m+n}\big), \qquad (2.1)$$

where $n \ge 1$, $m > 0$.

Proof Since $x_0 = y_0 = 0$, we have the following identity

$$x_i = \sum_{j=0}^{i-1} \Delta x_j, \quad y_i = \sum_{j=0}^{i-1} \Delta y_j, \quad i = 1, 2, \cdots, l. \qquad (2.2)$$

For $n = 1$, by (2.2), the left of inequality (2.1) can be written as

$$\sum_{i=1}^{l} \big(|x_i||\Delta y_i|^m + |y_i||\Delta x_i|^m\big) \le \sum_{i=1}^{l} \left[\sum_{j=0}^{i-1} |\Delta x_j||\Delta y_i|^m + \sum_{j=0}^{i-1} |\Delta y_j||\Delta x_i|^m\right]. \qquad (2.3)$$

For $n > 1$, using the Hölder's inequality with indices $\{\frac{n}{n-1}, n\}$, we have

$$\begin{aligned}
&\sum_{i=1}^{l} \big(|x_i|^n |\Delta y_i|^m + |y_i|^n |\Delta x_i|^m\big) \\
\le &\sum_{i=1}^{l} \left[\left(\sum_{j=0}^{i-1} |\Delta x_j|\right)^n |\Delta y_i|^m + \left(\sum_{j=0}^{i-1} |\Delta y_j|\right)^n |\Delta x_i|^m\right] \\
\le &\sum_{i=1}^{l} \left\{i^{n-1} \sum_{j=0}^{i-1} |\Delta x_j|^n |\Delta y_i|^m + i^{n-1} \sum_{j=0}^{i-1} |\Delta y_j|^n |\Delta x_i|^m\right\} \\
\le &\, l^{n-1} \sum_{i=1}^{l} \left\{\sum_{j=0}^{i-1} |\Delta x_j|^n |\Delta y_i|^m + \sum_{j=0}^{i-1} |\Delta y_j|^n |\Delta x_i|^m\right\}. \qquad (2.4)
\end{aligned}$$

From (2.3) and (2.4), for $n \ge 1$, we get that

$$\sum_{i=1}^{l} \big(|x_i|^n |\Delta y_i|^m + |y_i|^n |\Delta x_i|^m\big) \le l^{n-1} \sum_{i=1}^{l} \left\{\sum_{j=0}^{i-1} |\Delta x_j|^n |\Delta y_i|^m + \sum_{j=0}^{i-1} |\Delta y_j|^n |\Delta x_i|^m\right\}. \qquad (2.5)$$