Ann. of Appl. Math.
34:4(2018), 345-357

STABILITY AND BOUNDEDNESS OF SOLUTIONS FOR A CERTAIN FOURTH-ORDER DELAY DIFFERENTIAL EQUATION*†

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Abstract

In this paper, with the help of Lyapunov functional approach, sufficient conditions for the asymptotic stability of zero solution for a certain fourth-order non-linear delay differential equation are given.

Keywords Lyapunov functional approach; asymptotic stability; fourth-order non-linear delay differential equation

2000 Mathematics Subject Classification 34C10; 34A37

1 Introduction

As we know, the stability of zero solution plays an important role in the theory and applications of differential equations. In recent decades, by constructing Lyapunov functions or functionals many results have been obtained on the behaviour of solutions for higher order non-linear ordinary differential equations or higher order non-linear differential equations with delay, see [1-12, 14-16]. However, it is worth noting that there are only a few papers on the stability of solutions for certain fourth order non-linear differential equations with delay.

In [1], Sinha investigated the asymptotic behaviour of the solutions for a non-linear differential equations with finite lag

$$x^{(4)}(t) + f_1(\ddot{x}(t))\dddot{x}(t) + f_2(\dot{x}(t), \ddot{x}(t))\ddot{x}(t) + g(\dot{x}(t-r)) + h(x(t-r)) = 0. \quad (1.1)$$

In [2], Tunc considered a fourth-order non-linear delay differential equation

$$x^{(4)} + \varphi(\ddot{x})\dddot{x} + h(\dot{x})\ddot{x} + \phi(\dot{x}(t-r)) + f(x(t-r)) = 0 \quad (1.2)$$

and gave the asymptotic stability of solutions for (1.2).

*This research was supported by the National Natural Science Foundation of China (No.11671227).

†Manuscript received October 18, 2017; Revised October 12, 2018

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In [3], the authors studied the fourth-order delay differential equation of the form

$$x^{(4)} + \phi(\ddot{x})\dddot{x} + h(x, \dot{x}, \ddot{x}) + g(\dot{x}(t-r)) + f(x(t-r)) = 0 \tag{1.3}$$

and obtained sufficient conditions for the asymptotic stability of solutions.

In this paper, we investigate the same problem for the following fourth-order non-linear delay differential equation

$$\begin{aligned} &x^{(4)} + f_1(x, \dot{x}, \ddot{x}, \dddot{x}) + f_2(x, \dot{x}, \ddot{x}) + f_3(\dot{x}(t-r)) + f_4(x(t-r)) \\ =\, &p(t, x, \dot{x}, \ddot{x}, \dddot{x}, \dot{x}(t-r), x(t-r)) \end{aligned} \tag{1.4}$$

by constructing a new Lyapunov functional, where r is a positive constant, $f_1(x, \dot{x}, \ddot{x}, \dddot{x})$, $f_2(x, \dot{x}, \ddot{x})$, $f_3(\dot{x})$ and $f_4(x)$ are continuous differential functions and $f_2(0, 0, 0) = f_3(0) = f_4(0) = 0$, where we assume that the derivatives $\frac{\partial}{\partial x} f_2(x, y, z)$, $\frac{\partial}{\partial y} f_2(x, y, z)$, $f_3'(y)$ and $f_4'(x)$ exist and are continuous for all x, y, z. The dots indicate differentiation with respect to t.

2 Preliminaries

Before giving the main result of this paper we need the following definitions and stability criteria for the general autonomous delay differential system. We consider

$$\dot{x} = f(x_t), \quad x_t = x(t+\theta), \quad -r \le \theta \le 0,\ t \ge 0, \tag{2.1}$$

where $f : C_H \to \mathbf{R}^n$ is a continuous mapping with $f(0)=0$, $C_H = \{\phi \in C([-r, 0], \mathbf{R}^n) : \|\phi\| \le H\}$ and for $H_1 < H$, there exists an $L(H_1) > 0$, such that $|f(\phi)| \le L(H_1)$ when $||\phi|| \le H_1$, where H_1 and H are positive constants.

Definition 2.1[2-4] An element $\Psi \in C_H$ is in the ω-limit set of ϕ, say, $\Omega(\phi)$, if $x(t, 0, \phi)$ is defined on $[0, \infty)$ and there is a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$, such that $\|x_{t_n}(\phi) - \Psi\| \to 0$ as $n \to \infty$, where $x_{t_n}(\phi) = x(t_n + \theta, 0, \phi)$ for $-r \le \theta \le 0$.

Definition 2.2[2-4] A set $Q \subset C_H$ is an invariant set if for any $\phi \in Q$, the solution $x(t, 0, \phi)$ for (2.1) is defined on $[0, \infty)$, and $x_t(\phi) \in Q$ for $t \in [0, \infty)$.

Lemma 2.1[2-4] *If $\phi \in C_H$ satisfies that the solution $x_t(\phi)$ for* (2.1) *with $x_0(\phi) = \phi$ is defined on* $[0, \infty)$ *and $\|x_t(\phi)\| \le H_1 \le H$ for $t \in [0, \infty)$, then $\Omega(\phi)$ is a non-empty, compact, invariant set and* $\operatorname{dist}(x_t(\phi), \Omega(\phi)) \to 0$ *as $t \to \infty$.*

Lemma 2.2[2-4] *Let $V : C_H \to \mathbf{R}$ be a continuous functional satisfying a local Lipschitz condition, $V(0) = 0$ and that*

(i) $W_1(|\phi(0)|) \le V(\phi) \le W_2(\|\phi\|)$ *where $W_1(r)$ and $W_2(r)$ are wedges;*

(ii) $\dot{V}_{(2.1)}(\phi) \le 0$, *for $\phi \le C_H$.*

Then the zero solution for (2.1) *is uniformly stable. If we define $Z = \{\phi \in C_H :$*