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On the Distance Cospectrality of Threshold Graphs

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Abstract. A threshold graph can be represented as the binary sequence. In this paper, we present an explicit formula for computing the distance characteristic polynomial of a threshold graph from its binary sequence, and then give a necessary and sufficient condition to characterize two distance cospectral but non-isomorphic threshold graphs. As its applications, we obtain many families of distance cospectral threshold graphs. This provides a negative answer to the problem posed in [22].

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1 Introduction

Let Γ be a connected graph with vertex set $V(G)=\{v_1,v_2,\cdots,v_n\}$. The *distance* between v_i and v_j, denoted by $d(v_i,v_j)$ (or $d_{i,j}$ for short), is the length of a shortest path between v_i and v_j. Let $M=M(G)$ be a corresponding *graph matrix* defined in a prescribed way. The *M-polynomial* of G is defined as $P_G(x)=\det(\lambda I-M)$, where I is the identity matrix. The *M-spectrum* of G is a multiset consisting of the eigenvalues of its graph matrix M. It is well-known that there are several graph matrices including *adjacency matrix* $A(G)$, *Laplacian matrix* $L(G)$, *signless Laplacian matrix* $Q(G)$, distance matrix $\mathcal{D}(G)$ and so on.

A *threshold graph* is a graph with no induced subgraph isomorphic to the path of order 4, the cycle of order 4, or to two disjoint copies of K_2, the complete graph of order 2. Threshold graphs admit several equivalent definitions, in particular, a recursive definition based on a binary code will be relevant to this paper, and will be described here.

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Recall that a vertex is *isolated* if it has no neighbors, and is *dominating* if it is adjacent to all other vertices. Then, another equivalent definition of a threshold graph is that it can be constructed from the one-vertex graph by repeatedly adding an isolated vertex or a dominating vertex. Hence, for a threshold graph Γ with $V(\Gamma)=\{v_1,\cdots,v_n\}$, we can use a $\{0,1\}$-sequence $\mathbf{b}=(b_1,\cdots,b_n)$ to represent it, where $b_i=0$ if v_i is an isolated vertex and $b_i=1$ if v_i is a dominating vertex. As usual, we set $b_1=0$. Therefore, the sequence $\mathbf{b}$, named by *representation sequence*, can be written as $\mathbf{b}=(0^{b_1},1^{b_2},\cdots,0^{b_{n-1}},1^{b_m})$, where $b_i\geq 1$ $(1\leq i\leq m)$. Obviously, Γ is connected if and only if $b_n=1$. In what follows, we only consider the connected threshold graphs.

Threshold graphs were first introduced by Chvátal and Hammer [7] and Henderson and Zalcstein [14] in 1977. Threshold graphs have received a lot of attentions because of their numerous applications in compute science and psychology [24]. And also they have nice spectral properties with respect to the adjacency, Laplacian, and normalized Laplacian matrix, we turn the readers to [3–5, 13, 15–18, 25].

In this paper, we pay attention to the $\mathcal{D}$-spectra of threshold graphs. The *distance matrix* of Γ is the $n\times n$ matrix whose (i,j)-entry is equal to $d(v_i,v_j)$, for $1\leq i,j\leq n$. Remarkably, it becomes a hot topic to study the eigenvalues of the distance matrix of a graph ever since the appearance of the paper [11] by Graham and Pollack, which established a relationship between the number of negative eigenvalues of the distance matrix and the addressing problem in data communication systems. For more details about the distance matrix, we refer the readers to the papers [1, 2, 20, 23] for examples.

We need several extra notation and terminology. Graphs with the same M-spectrum are called *M-cospectral graphs*. A graph G is said to be *determined by its M-spectrum* if there is no other non-isomorphic graph with the same M-spectrum. The background of this problem "which graphs are determined by their spectrum?" originates from Chemistry (in 1956, Günthadr and Primas [12] raised this question in the context of Hückel's theory). For additional remarks on this topic we refer the readers to see the excellent surveys [8,9]. In the beginning, the researchers concentrate on constructing cospectral graphs, such as Godsil-Mckay Switching [10] and Shwenk's method [26]. Recently, some researchers discussed the cospectrality of the threshold graphs. Carvalho et al. [6] showed that 2^{n-4} threshold graphs with order n have Q-cospectral threshold graphs. Very lately, Lazzarin, Márquez and Tura [19] proved that no threshold graphs are A-cospectral. Additionally, Lu, Huang and Lou [22] put forward the following problem:

Problem 1. Whether threshold graphs are determined by their distance spectra?

In this paper, we will focus on this problem and give a negative answer to the above problem. The paper is organized as follows: In Section 2, we present some basic lemmas and results. In Section 3, we determine the $\mathcal{D}$-polynomial of threshold graphs. In Section 4, we give a necessary and sufficient condition of $\mathcal{D}$-cospectral threshold graphs and provided the infinite classes of $\mathcal{D}$-cospectral threshold graphs with fixed binary sequences. At last, we propose a related problem for further study.