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A New Discrete Energy Technique for Multi-Step Backward Difference Formulas

Hong-Lin Liao[1,2], Tao Tang[3,4] and Tao Zhou[5,*]

[1] *Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, P.R. China.*
[2] *Key Laboratory of Mathematical Modelling and High Performance Computing of Air Vehicles (NUAA), MIIT, Nanjing 211106, P.R. China.*
[3] *Division of Science and Technology, BNU-HKBU United International College, Zhuhai, Guangdong, P.R. China.*
[4] *International Center for Mathematics, Southern University of Science and Technology, Shenzhen, Guangdong, P.R. China.*
[5] *LSEC, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, P.R. China.*

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Abstract. The backward differentiation formula (BDF) is a popular family of implicit methods for the numerical integration of stiff differential equations. It is well noticed that the stability and convergence of the A-stable BDF1 and BDF2 schemes for parabolic equations can be directly established by using the standard discrete energy analysis. However, such classical analysis seems not directly applicable to the BDF-$\mathtt{k}$ with $3 \leq \mathtt{k} \leq 5$. To overcome the difficulty, a powerful analysis tool based on the Nevanlinna-Odeh multiplier technique [Numer. Funct. Anal. Optim., 3:377-423, 1981] was developed by Lubich et al. [IMA J. Numer. Anal., 33:1365-1385, 2013]. In this work, by using the so-called discrete orthogonal convolution kernel technique, we recover the classical energy analysis so that the stability and convergence of the BDF-$\mathtt{k}$ with $3 \leq \mathtt{k} \leq 5$ can be established.

AMS subject classifications: 65M06, 65M12

Key words: Linear diffusion equations, backward differentiation formulas, discrete orthogonal convolution kernels, positive definiteness, stability and convergence.

1 Introduction

The backward differentiation formula (BDF) is a family of implicit methods for the numerical integration of stiff differential equations [6, 11, 12]. They are linear multistep

*Corresponding author. *Email addresses:* `liaohl@nuaa.edu.cn`, `liaohl@csrc.ac.cn` (H.-L. Liao), `ttang@uic.edu.cn` (T. Tang), `tzhou@lsec.cc.ac.cn` (T. Zhou)

methods that approximate the derivative of the unknown function using information from already computed time points, thereby increasing the accuracy of the approximation. These methods are especially used for the solution of stiff differential equations whose numerical stability is indicated by the region of absolute stability. More precisely, if the region of stability contains the left half of the complex plane, then the method is said to be A-stable. However, it is known that backward differentiation methods with an order higher than 2 cannot be A-stable, i.e., only the first-order and second-order backward differentiation formulas (i.e., BDF1 and BDF2) are A-stable. For parabolic equations, it is also well-known that the energy stability and convergence of A-stable BDF1 and BDF2 methods can be established by using the standard discrete energy analysis, see, e.g., [12]. However, this standard analysis technique is not directly applicable to higher order BDF schemes. This results in several remedies to recover the L^2-norm stability and convergence for the non-A-stable $\mathtt{k}$-step backward difference formulas with $3\le \mathtt{k}\le 5$. It is particularly noted that due to the seminal work of Lubich et al. [10], the Nevanlinna-Odeh multiplier technique [11] has been successfully used for this purpose, see e.g. [1–3] and references therein. The key idea of such a multiplier technique relies on the equivalence between A-stability and G-stability of Dahlquist [4]. Another useful tool for the numerical analysis of BDF-$\mathtt{k}$ schemes is the telescope formulas by Liu [9], which is also based on the Dahlquist G-stability theory [4].

We have a natural question: is there a straightforward discrete energy analysis for the BDF-$\mathtt{k}$ with $3\le \mathtt{k}\le 5$? The aim of this work is to provide a definite answer by introducing a novel yet straightforward discrete energy method based on the so-called discrete orthogonal convolution (DOC) kernel technique [8]. To this end, we consider the linear reaction-diffusion problem in a bounded convex domain Ω,

$$\partial_t u-\varepsilon\Delta u=\beta(x,t)u+f(t,x),\quad x\in\Omega,\quad 0<t<T, \tag{1.1}$$

subject to the Dirichlet boundary condition $u=0$ on a smooth boundary $\partial\Omega$. The initial data is set to be $u(0,x)=u_0(x)$. We assume that the diffusive coefficient $\varepsilon>0$ is a constant and the reaction coefficient $\beta(x,t)$ is smooth and bounded by $\beta^*>0$.

Let $t_k=k\tau$ be a uniform discrete time-step with $\tau:=T/N$. For any discrete time sequence $\{v^n\}_{n=0}^N$, we denote

$$\nabla_\tau v^n:=v^n-v^{n-1},\quad \partial_\tau v^n:=\nabla_\tau v^n/\tau.$$

For a fixed index $3\le\mathtt{k}\le 6$, we shall view the BDF-$\mathtt{k}$ formula $D_{\mathtt{k}}v^n$ as a discrete convolution summation as follows

$$D_{\mathtt{k}}v^n:=\frac{1}{\tau}\sum_{k=1}^{n}b_{n-k}^{(\mathtt{k})}\nabla_\tau v^k,\quad n\ge\mathtt{k}, \tag{1.2}$$

where the associated BDF-$\mathtt{k}$ kernels $b_j^{(\mathtt{k})}$ (vanish if $j\ge\mathtt{k}$, see Table 1) are generated by

$$\sum_{\ell=1}^{\mathtt{k}}\frac{1}{\ell}(1-\zeta)^{\ell-1}=\sum_{\ell=0}^{\mathtt{k}-1}b_\ell^{(\mathtt{k})}\zeta^\ell,\quad 3\le\mathtt{k}\le 6. \tag{1.3}$$