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ERROR ESTIMATES AND BLOW-UP ANALYSIS OF A FINITE-ELEMENT APPROXIMATION FOR THE PARABOLIC-ELLIPTIC KELLER-SEGEL SYSTEM

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Abstract. The Keller-Segel equations are widely used for describing chemotaxis in biology. Recently, a new fully discrete scheme for this model was proposed in [46], mass conservation, positivity and energy decay were proved for the proposed scheme, which are important properties of the original system. In this paper, we establish the error estimates of this scheme. Then, based on the error estimates, we derive the finite-time blowup of nonradial numerical solutions under some conditions on the mass and the moment of the initial data.

Key words. Parabolic-elliptic systems, finite element method, error estimates, finite-time blowup.

1. Introduction.

Keller and Segel first proposed a nonlinear model in the 1970s to describe the effect of cell aggregation in [27, 28]. A simplified Keller-Segel model in 2-D is given by

$$\frac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v), \quad x \in \Omega,\ t > 0, \tag{1}$$

$$0 = \Delta v - v + \alpha u, \quad x \in \Omega,\ t > 0, \tag{2}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$. The unknown $u = u(x,t)$ and $v = v(x,t)$ represent the concentration of the organism and chemoattractant respectively. The parameters χ, α are positive constants with χ being the sensitivity of chemotaxis. The model is supplemented with initial conditions

$$u(x, t=0) = u_0(x),\ v(x, t=0) = v_0(x),\ x \in \Omega,$$

and no flux boundary conditions

$$\frac{\partial u}{\partial \boldsymbol{n}} - \chi u \frac{\partial v}{\partial \boldsymbol{n}} = 0,\ \frac{\partial v}{\partial \boldsymbol{n}} = 0, \quad x \in \partial\Omega,\ t > 0,$$

where $\boldsymbol{n}$ denotes the unit outward normal vector to the boundary $\partial\Omega$, $\partial/\partial\boldsymbol{n}$ represents differentiation along $\boldsymbol{n}$ on $\partial\Omega$.

A different version of the Keller-Segel model consists in replacing (2) by

$$\frac{\partial v}{\partial t} = \Delta v - v + \alpha u, \quad x \in \Omega,\ t > 0. \tag{3}$$

The equation (1) describes the motion of the organism u. The term $F = -\nabla u + \chi u \nabla v$ is the flux, and the effect of diffusion $-\Delta u$ and that of chemotaxis $\chi \nabla \cdot (u \nabla v)$ are competing for u to vary. The equation (2) describes the change in concentration of the chemoattractant v, it is influenced by the diffusion and the decay of the chemoattractant as well as the growth of the organism. In general, the chemoattractant particles are much smaller than the organism particles, thus it diffuses faster, which means that the diffusion of the chemoattractant will reach the

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equilibrium state in a relatively short time. The model (1)-(2) is called parabolic-elliptic system. On the other hand, (1) with (3) is a parabolic-parabolic system.

The solution of the Keller-Segel model (1)-(2) has several well-known properties, particularly, it may blow up in finite time. Various aspects and results for the classical Keller-Segel model since 1970, along with some open questions, are summarized in [25]. Positivity, mass conservation and energy dissipation of Keller-Segel equations can be found in [35],[36],[47],[29] and [6], which plays an important role to study the Keller-Segel system. Blanchet, Dolbeault and Perthame presented in [3] a detail proof of the existence of weak solutions when the initial mass is below the critical mass, above which any solution to the parabolic-elliptic systems blows up in finite time in the whole Euclidean space. In [37], Nagai demonstrated the finite-time blowup of nonradial solutions under some assumptions on the mass and the moment of the initial data. As for the parabolic-parabolic systems, Blanchet proved in [2] the optimal critical mass of the solutions in $\mathbb{R}^d$ with $d \geq 3$. Wei proved that for every nonnegative initial data in $L^1(R^2)$, the 2-D Keller-Segel equation is globally well-posed if and only if the total mass $M \leq 8\pi$ in [49].

Although the large time behavior of the solution of the Keller-Segel model (1)-(2) has been well studied, there is still much to explore on the numerical side. Since the Keller-Segel equations possess three important properties: positivity, mass conservation and energy dissipation, it is preferable that numerical schemes can preserve these properties at the discrete level. In [26], the existence of weak solutions and upper bounds for the blow-up time for time-discrete (including the implicit Euler, BDF and Runge-Kutta methods) approximations of the parabolic-elliptic Keller-Segel models in the two-dimensional whole space are established. Liu, Li and Zhou proposed a numerical method in [34] which preserves both positivity and asymptotic limit, the proposed numerical method does not generate negative density if initialized properly under a less strict stability condition. Saito and Suzuki presented a finite difference scheme in [42] which satisfies the conservation of a discrete L^1 norm.

Some finite element methods are proposed in previous works. Saito presented a finite element scheme for parabolic-elliptic systems in [43] that satisfies both positivity and mass conservation properties. Under some assumptions on the regularity of solutions, the error estimates were established. Saito further constructed the finite element methods to the parabolic-parabolic systems in [44] and derived error analysis by using analytical semigroup theory. Gurusamy and Balachandran proposed a finite element method for parabolic-parabolic systems and established the existence of approximate solutions by using Schauder's fixed point theorem in [23]. Further the error estimates for the approximate solutions in H^1-norm were derived.

The discontinuous Galerkin methods can be also used to solve the Keller-Segel equations. Epshteyn and Kurganov developed a family of new interior penalty discontinuous Galerkin methods and proved error estimates for the proposed high-order discontinuous Galerkin methods in [15]. Epshteyn and Izmirlioglu further constructed a discontinuous Galerkin method for Keller-Segel model in [16] and obtained fully discrete error estimates for the proposed scheme. In 2017, Li, Shu and Yang applied the local discontinuous Galerkin (LDG) method to 2D Keller-Segel chemotaxis model in [30], they improved the results upon [15] and gave optimal rate of convergence under special finite element spaces before the blow-up occurs. In 2019, Guo, Li and Yang constructed a consistent numerical energy and prove the energy dissipation with the LDG discretization in [22].