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STRUCTURE-PRESERVING NUMERICAL METHODS FOR A CLASS OF STOCHASTIC POISSON SYSTEMS

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Abstract. We propose a type of numerical methods for a class of stochastic Poisson systems with invariant energy. The proposed numerical methods preserve both the energy and the Casimir functions of the systems. In addition, we provide a new approach of constructing stochastic Poisson integrators which respect the Poisson structure and the Casimir functions of stochastic Poisson systems based on coordinate transformations on the midpoint method. Numerical tests are performed to demonstrate our theoretical analysis.

Key words. stochastic Poisson systems, structure-preserving algorithms, Poisson structure, Casimir functions, Poisson integrators.

1. Introduction

Stochastic Poisson systems (SPSs) are generalizations of stochastic Hamiltonian systems ([2, 9, 19]) and have the following form ([9]):

$$d\boldsymbol{y}(t) = \boldsymbol{B}(\boldsymbol{y}(t)) \left(\nabla H_0(\boldsymbol{y}(t))dt + \sum_{r=1}^{s} \nabla H_r(\boldsymbol{y}(t)) \circ dW_r(t) \right),$$

$$\boldsymbol{y}(0) = \boldsymbol{y}_0, \tag{1}$$

where $\boldsymbol{y} = (y^1, \ldots, y^m)^T \in \mathbb{R}^m$, $H_r(\boldsymbol{y})$ $(r = 0, \ldots, s)$ are smooth functions of $\boldsymbol{y}$, $\boldsymbol{\mathcal{W}}(t) := (W_1(t), \ldots, W_s(t))$ is an s-dimensional standard Wiener process defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$, '$\circ$' denotes the Stratonovich differential, and $\boldsymbol{B}(\boldsymbol{y}) = (b_{ij}(\boldsymbol{y}))$ is a smooth $m \times m$ matrix–valued function of $\boldsymbol{y}$ which is skew-symmetric $(b_{ij}(\boldsymbol{y}) = -b_{ji}(\boldsymbol{y}))$ and satisfies

$$\sum_{l=1}^{m} \left(\frac{\partial b_{ij}(\boldsymbol{y})}{\partial y^l} b_{lk}(\boldsymbol{y}) + \frac{\partial b_{jk}(\boldsymbol{y})}{\partial y^l} b_{li}(\boldsymbol{y}) + \frac{\partial b_{ki}(\boldsymbol{y})}{\partial y^l} b_{lj}(\boldsymbol{y}) \right) = 0, \tag{2}$$

for all $i, j, k \in \{1, \ldots, m\}$.

If the dimension $m = 2d$ is an even integer, and

$$\boldsymbol{B}(\boldsymbol{y}) \equiv \boldsymbol{J}^{-1} = \begin{pmatrix} \boldsymbol{0}_d & -\boldsymbol{I}_d \\ \boldsymbol{I}_d & \boldsymbol{0}_d \end{pmatrix}$$

where $\boldsymbol{I}_d$ denotes the d-dimensional identity matrix, then SPSs (1) degenerate to stochastic Hamiltonian systems ([17, 18, 19]). It was proved in [9] that almost surely the phase flow of a SPS $\boldsymbol{\varphi}_t : \boldsymbol{y} \to \boldsymbol{\varphi}_t(\boldsymbol{y})$ possesses the Poisson structure:

$$\frac{\partial \boldsymbol{\varphi}_t(\boldsymbol{y})}{\partial \boldsymbol{y}} \boldsymbol{B}(\boldsymbol{y}) \frac{\partial \boldsymbol{\varphi}_t(\boldsymbol{y})}{\partial \boldsymbol{y}}^T = \boldsymbol{B}(\boldsymbol{\varphi}_t(\boldsymbol{y})), \quad \forall t \geq 0, \quad a.s. \tag{3}$$

Moreover, if the rank of $\boldsymbol{B}(\boldsymbol{y})$ is not full, there exists at least one Casimir function $C(\boldsymbol{y})$ with the property $\nabla C(\boldsymbol{y})^T \boldsymbol{B}(\boldsymbol{y}) \equiv \boldsymbol{0}$ $(\forall \boldsymbol{y})$ ([7]). Casimir functions are invariants of the SPSs almost surely ([9]), i.e. $C(\boldsymbol{y}(t)) \equiv C(\boldsymbol{y}_0)$ along the solution $\boldsymbol{y}(t)$

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of (1) $\forall t \geq 0$ almost surely, since

$$\begin{aligned} dC(\boldsymbol{y}) =& \nabla C(\boldsymbol{y})^T d\boldsymbol{y} \\ =& \nabla C(\boldsymbol{y})^T \boldsymbol{B}(\boldsymbol{y}) \left(\nabla H_0(\boldsymbol{y})dt + \sum_{r=1}^{s} \nabla H_r(\boldsymbol{y}) \circ dW_r(t) \right) \\ =& 0. \end{aligned}$$

A numerical method $\{\boldsymbol{y}_n : n \in \mathbb{N}\}$ of (1) is said to preserve the Casimir function $C(\boldsymbol{y})$ if

$$C(\boldsymbol{y}_{n+1}) = C(\boldsymbol{y}_n), \quad \forall n \in \mathbb{N}, \text{ a.s..}$$

It is not difficult to see that, for any $i \in \{0, \dots, s\}$, if

$$\{H_i(\boldsymbol{y}), H_j(\boldsymbol{y})\} := \nabla H_i(\boldsymbol{y})^T \boldsymbol{B}(\boldsymbol{y}) \nabla H_j(\boldsymbol{y}) = 0 \text{ for all } j = 0, \dots, s \text{ and all } \boldsymbol{y},$$

where $\{H_i(\boldsymbol{y}), H_j(\boldsymbol{y})\}$ is called the Poisson bracket of $H_i(\boldsymbol{y})$ and $H_j(\boldsymbol{y})$, then

$$dH_i(\boldsymbol{y}) = \nabla H_i(\boldsymbol{y})^T d\boldsymbol{y} = 0.$$

In this case $H_i(\boldsymbol{y})$ is an invariant Hamiltonian of (1). When $H_r \equiv 0$ for $r = 1, \dots, s$, SPSs (1) degenerate to deterministic Poisson systems ([7, 13]).

Poisson systems find applications in many scientific and engineering areas such as astronomy, robotics, quantum mechanics, electrodynamics and so on ([31]). Given the characterization of the Poisson structure (3), a numerical method $\{\boldsymbol{y}_n : n \in \mathbb{N}\}$ is said to preserve the Poisson structure of the system if it satisfies (see e.g. [7, 9])

$$\frac{\partial \boldsymbol{y}_{n+1}}{\partial \boldsymbol{y}_n} \boldsymbol{B}(\boldsymbol{y}_n) \frac{\partial \boldsymbol{y}_{n+1}}{\partial \boldsymbol{y}_n}^T = \boldsymbol{B}(\boldsymbol{y}_{n+1}) \text{ (a.s. in stochastic cases)}, \; n \in \mathbb{N}. \tag{4}$$

Numerical methods for Poisson systems that can preserve both the Poisson structure and the Casimir functions are called Poisson integrators. Even for deterministic Poisson systems, it is challenging to construct general Poisson integrators in case the structure matrix $\boldsymbol{B}(\boldsymbol{y})$ is nonconstant (p. 270 of [7], [10]). During the last decades, there have been numerous studies exploiting special structures of particular deterministic Poisson systems to construct Poisson integrators or other structure-preserving numerical methods for them. Such methods have been shown to produce much better long-time numerical behavior than other general-purpose methods (see e.g. [1, 3, 4, 5, 6, 7, 15, 16, 21, 22, 23, 24, 28, 29, 30] and references therein).

Stochastic Poisson systems were recently proposed and numerically studied (see e.g. [2, 8, 9, 12, 25, 26]), where stochastic Poisson integrators or energy (Hamiltonian)-preserving methods were investigated. For the following stochastic Poisson system

$$d\boldsymbol{y}(t) = \boldsymbol{B}(\boldsymbol{y}(t)) \nabla H(\boldsymbol{y}(t))(dt + c \circ dW(t)), \tag{5}$$

where $\boldsymbol{B}(\boldsymbol{y})$ is a skew-symmetric matrix satisfying (2) and c is a non-zero constant, $H(\boldsymbol{y})$ is obviously an invariant Hamiltonian and called the energy of the system ([2]). [2] proposed a class of numerical methods that can preserve the energy $H(\boldsymbol{y})$ and quadratic Casimir functions of the system. [12] constructed a class of explicit parametric stochastic Runge–Kutta methods which preserve the energy $H(\boldsymbol{y})$ for suitable parameters and can achieve any prescribed mean-square orders.

In (5), when $c = 0$ and

$$\begin{aligned} \boldsymbol{B}(\boldsymbol{y}) &= \left(b_{ij}^0 y^i y^j\right) = diag(y^1, \dots, y^m) \boldsymbol{B_0} \, diag(y^1, \dots, y^m), \\ H(\boldsymbol{y}) &= \sum_{i=1}^{m} \beta_i y^i - p_i \ln y^i, \end{aligned} \tag{6}$$