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AN EXPONENTIAL TIME DIFFERENCING TIME-STEPPING SCHEME FOR THE TRACER EQUATIONS IN MPAS-OCEAN

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Abstract. Exponential time differencing (ETD) methods, also known as exponential integrators, constitute a class of numerical methods for the time integration of stiff systems of differential equations. This manuscript investigates an ETD scheme for solving the tracer equations appearing in primitive equation ocean models, and shows the results obtained when such a scheme is applied within a full ocean circulation model. The main idea behind the scheme is the treatment of the vertical terms (transport and diffusion) with a matrix exponential, whereas the horizontal terms are dealt with in an explicit way. The performance of the ETD scheme is compared against that of other semi-implicit time-stepping schemes for realistic ocean configurations on quasi-uniform and variable resolution meshes.

Key words. Exponential time differencing, tracer equation, MPAS-Ocean.

1. Introduction

In ocean modeling, the tracer equation describes the transport of tracers, like temperature, salinity, radioactive material (tritium), chemical concentrations (chlorofluorocarbons) and biological organisms (phytoplankton). In a modern ocean circulation model (OCM), 2 to 50 tracers are tracked in a realistic simulation meaning that 2 to 50 tracers equations have to be solved, causing a significant computational load. Hence, efficiently solving multiple tracer equations is an important task in an ocean model.

Tracer evolution is described by an advection-diffusion equation of the form,

$$\partial_t T + \nabla_x \cdot (uT) + \mathcal{D}_x T + \partial_z (wT) - \partial_z(\kappa_z \partial_z T) = q(T), \tag{1}$$

where T is the tracer concentration, $\mathbf{u} = (u, w) \in R^3$ is the velocity of water, which is split into the horizontal velocity $u \in R^2$ and the vertical velocity w, $\mathcal{D}_x$ is the horizontal diffusion operator, κ_z is the vertical diffusion coefficient, and $q(T)$ represents interior sources or sinks. Models like POP [8], MITgcm [9] and MPAS-Ocean [2] use semi-implicit time-stepping schemes for advancing the tracer equations in time, since fully explicit schemes would be bound by severe time-step restrictions associated with fast mixing processes. Tracer vertical mixing usually occurs on fast time-scales and can be induced by density differences and/or by turbulent motions. These fast processes are often represented as a very strong diffusion coefficient κ_z in the vertical diffusion term $\partial_z(\kappa_z \partial_z T)$, resulting in very short time steps when using a fully explicit time integrator. For this reason, in POP, MITgcm and MPAS-Ocean, vertical diffusion can be treated implicitly with an implicit Euler algorithm. The remaining terms of the tracer equation (horizontal and vertical advection and horizontal diffusion) are treated explicitly in all the three models and the integration can proceed with longer time steps, typically at the horizontal advective CFL timescale.

However, the explicit treatment of vertical advection may not be optimal when many vertical layers are used to resolve mixed layer processes near the surface. For

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example, [2] employs 1-15km horizontal mesh spacing, but a much smaller 10 m mesh spacing in the vertical. This spacing, combined with higher surface velocities can lead to small CFL constraints and an implicit treatment of vertical advection can be beneficial. Implicit-explicit (IMEX) schemes [26, 27, 28] are indeed widely used in modern high-resolution numerical weather prediction and climate models. These schemes use an implicit component to deal with the terms in the model equations describing the fastest moving waves, allowing for longer time steps than fully explicit methods. An alternative to employing an implicit step for the fastest moving waves is the use of a matrix exponential. In [10], a time-stepping scheme that treats vertical diffusion and vertical advection in an exact way with a matrix exponential has been used to solve the tracer equation on idealized test cases. This scheme consists of an exponential time differencing (ETD) method where all the vertical terms are treated with a matrix exponential, whereas the horizontal terms are dealt with in an explicit way. ETD methods, also known as exponential integrators, have recently gained attention in the atmosphere and ocean modeling community due to their stability properties that allow time-steps considerably larger than those dictated by the CFL condition [11, 12, 13, 14, 15, 16, 17]. For a review of exponential integrators we refer to [6].

In this work, we consider the ETD time-stepping scheme presented in [10], called hereafter ETD-CPG, and study its performance in MPAS-Ocean on realistic ocean test cases. The operator splitting used by ETD-CPG (vertical terms treated with a matrix exponential vs horizontal terms treated explicitly) has two main advantages. First, higher accuracy is expected compared to other semi-implicit methods for the tracer equation because of an exact treatment of the fast vertical terms. Second, a focus on the vertical terms simplifies the implementation and reduces the cost in a parallel computing context. Ocean models decompose the domain only in the horizontal so computation of vertical terms can take place solely within a node and no additional communication is needed for a parallel implementation. The addition of accelerators like Graphic Processing Units (GPUs) within a node can be used to further optimize and reduce the cost of the implementation. Exploiting this local implementation, we use an approach based on scaling and squaring relations [18, 19] for the computation of the matrix functions. This approach, already presented in [10], is based on polynomials of moderate degree and results in a consistent and stable approximation of the φ-functions described below.

ETD-CPG has already been tested on 2D idealized test cases for the tracer equation, where it was shown that larger time-steps could be taken than other semi-implicit time-stepping schemes. Since we are going to test ETD-CPG on realistic ocean test cases, the time step used will be dictated by the dynamics, i.e. the change in time of the velocity and thickness of the vertical layers. In the tests performed in this work, we use both quasi-uniform and variable-resolution meshes on a sphere and the initial conditions are interpolated from the data gathered by the Polar Science Center Hydrographic Climatology [23].

The paper is organized as follows. Section 2 describes the vertical and horizontal discretization of the tracer equation in MPAS-Ocean. Section 3 describes the ETD-CPG method, focusing on the structure of the linear operator and the computation of the matrix functions. In section 4 numerical results are presented showing the convergence of the method and its performance on two test cases with realistic ocean configurations. Finally, in section 5 we draw our conclusions.