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# Completion of the Guo-Hierarchy Integrable Coupling with Self-Consistent Sources in a Nonlinear Wave System

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**Abstract.** Water waves are actively studied. A new method to generate new wave systems through making perturbation in matrix spectral problems for integrable couplings is presented, which is called the "completion process of integrable couplings". As its application, we construct an integrable coupling hierarchy and show that each equation in the resulting hierarchy has a bi-Hamiltonian structure by taking use of the component-trace identity. Moreover, the self-consistent sources of integrable coupling is presented based on the theory of self-consistent sources.

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**Key words**: Integrable coupling, Guo-hierarchy, bi-Hamiltonian structure, self-consistent source.

## 1. Introduction

Water waves have been actively studied [5,6,29,43]. For the Earth, water is at the core of sustainable development and at the heart of adaptation to climate change. The finding of new integrable couplings has become an important area of research in mathematical physics [11–18,20,23,25,30,45,46]. Integrable couplings are coupled systems of integrable equations, which has been introduced when we study of Virasoro symmetric algebras. It is an important topic to look for integrable couplings because integrable couplings have

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much richer mathematical structures and better physical meanings. Integrable couplings are coupled systems which contain given integrable equations as their sub-systems [14]. Suppose $u_t = K(u)$ is a given integrable system, its integrable coupling is an enlarged triangular integrable system of the following form:

$$\begin{aligned} u_t &= K(u), \\ v_t &= S(u,v). \end{aligned}$$

Integrable couplings usually show various specific mathematical structures, such as block matrix type Lax representations, infinitely many symmetries, bi-Hamiltonian structures and conservation laws of triangular form [17, 18, 46].

Soliton equations with self-consistent sources has been paid considerable attention in recent years. Physically, the sources may result in solitary waves with a non-constant velocity and therefore lead to a variety of dynamics of physical models. For example, the nonlinear Schrödinger equation with self-consistent sources is relevant to some problems of plasma physics and solid state physics [2, 3, 26]. There are many methods to get exact solutions of soliton equations with self-consistent sources, such as $\partial$-method and gauge transformation [27], and inverse scattering transform method [26]. Recently, modified discrete KP equation, the generalized super-NLS-mKdV hierarchy, and variable coefficient super AKNS hierarchy were presented [9, 22, 36, 37].

An important example of integrable couplings is the first-order perturbation system [20]

$$\begin{aligned} u_t &= K(u), \\ v_t &= K^{'}(u)[v], \end{aligned}$$

where $K^{'}(u)[v]$ denotes the Gateaux derivative

$$K^{'}(u)[v] = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} K(u+\varepsilon v, u_x+\varepsilon v_x, \cdots).$$

An arbitrary Lie algebra over a field of characteristic zero has a semi-direct sum structure of a solvable Lie algebra and a semisimple Lie algebra, which is stated by the Levi-Mal'tsev theorem. So, zero curvature equations over semi-direct sums of Lie algebras, i.e., non-semisimple Lie algebras, lay the foundation for generating integrable couplings.

A simple non-semisimple Lie algebra $\bar{\mathfrak{g}}$ consists of square matrices of the following block form [15]:

$$M(A_1,A_2) = \begin{pmatrix} A_1 & A_2 \\ 0 & A_1 \end{pmatrix},$$

$A_1$ and $A_2$ are two arbitrary square matrices of the same order, two subalgebras $\tilde{\mathfrak{g}} = \{M(A_1,0)\}$ and $\tilde{\mathfrak{g}}_c = \{M(0,A_2)\}$ which form a semi-direct sum: $\bar{\mathfrak{g}} = \tilde{\mathfrak{g}} \uplus \tilde{\mathfrak{g}}_c$. The notion of semidirect sums means that the two subalgebras $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}_c$ satisfy $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_c] \subseteq \tilde{\mathfrak{g}}_c$, which require the closure property between and under the matrix multiplication: $\tilde{\mathfrak{g}}\tilde{\mathfrak{g}}_c, \tilde{\mathfrak{g}}_c\tilde{\mathfrak{g}} \subseteq \tilde{\mathfrak{g}}_c$. The steps of the scheme for building integrable coupling associated with $\bar{\mathfrak{g}}$ is now presented briefly as follows: