Some Elementary Results Related to the Cauchy's Mean Value Theorem

German Lozada-Cruz
e-mail: german.lozada@unesp.br
Departamento de Matemática
Instituto de Biociências, Letras e Ciências Exatas (IBILCE)
Universidade Estadual Paulista (UNESP)
15054-000 São José do Rio Preto, São Paulo, Brazil

Abstract

In this note we prove some elementary results of Cauchy's mean value theorem. The main tools employed to get these are auxiliary functions.

1 Introduction

We know that mean value theorems are important tools in real analysis. The first one that we learn is the famous Lagrange's mean value theorem ([2, Theorem 2.3] or [10, Theorem 4.12] e.g.) and it asserts that a function $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists $\eta \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(\eta). \tag{1}$$

This mean value theorem is used to solve a great variety of problems in optimization, economics, etc.

If $f(a) = f(b)$ in (1), then the Lagrange's mean value theorem reduces to Rolle's theorem ([10, Theorem 4.11]). The equivalence between Rolle's and Lagrange's mean value theorems has been proved for example in [11, Theorem B].

In 1958, T. M. Flett [1] proved a variant of Lagrange's mean value theorem. Other authors obtained variants of Lagrange's mean value theorem (see [14] and [4] for example).

In 1977, R. E. Myers [9] proved that there are nine possible quotients in (1) having one of the values $f(b) - f(a)$, $f(\eta) - f(a)$, $f(b) - f(\eta)$ for numerators, and one of $b - a$, $\eta - a$, $b - \eta$ for denominators.

The second mean value theorem is the Cauchy's mean value theorem ([10, Theorem 4.14], [12, Theorem 2.17]), which is a generalization of the Lagrange's mean value theorem. It establishes the relationship between the derivatives of two functions and the variation of these functions on a finite interval.

Theorem 1 (Cauchy's Mean Value Theorem) *Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions on $[a, b]$, differentiable on (a, b) and $g'(x) \neq 0$ for all $x \in (a, b)$. Then, there exists $\eta \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\eta)}{g'(\eta)}. \tag{2}$$

For a geometric interpretation of Cauchy's mean value theorem consider the curve $\gamma : [a, b] \to \mathbb{R}^2$ given by $\gamma(t) = (f(t), g(t))$. According to the Cauchy's mean value theorem, there is a point $C = (f(\eta), g(\eta))$ on the curve γ where the tangent is parallel to the chord joining the points $A = (f(a), g(a))$ and $B = (f(b), g(b))$ of the curve (see Figure 1).

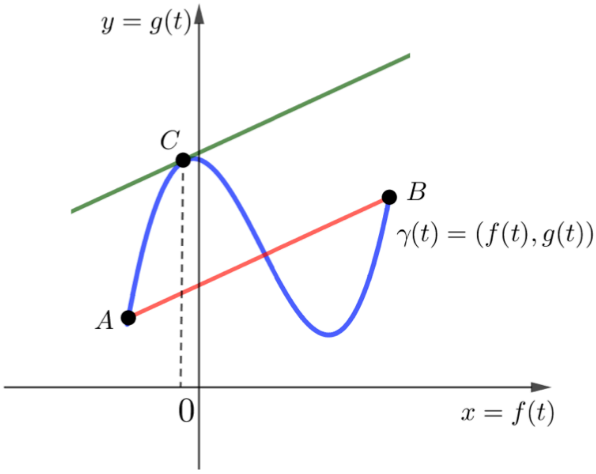

Figure 1: Geometric interpretation of Cauchy's mean value theorem

W.-C. Yang used technological tools and gave a geometric interpretation of Cauchy's mean value theorem (see [15]).

In 2000, E. Wachnicki (see [16, Theorem 1.3]) proved the following variant of Cauchy's mean value theorem.

Theorem 2 (Wachnicki's Theorem) *Let $f, g : [a, b] \to \mathbb{R}$ be differentiable functions on $[a, b]$. Suppose that $g'(x) \neq 0$ for all $x \in [a, b]$ and*

$$\frac{f'(a)}{g'(a)} = \frac{f'(b)}{g'(b)}. \tag{3}$$

Then, there exists $\eta \in (a, b)$ such that

$$\frac{f(\eta) - f(a)}{g(\eta) - g(a)} = \frac{f'(\eta)}{g'(\eta)}. \tag{4}$$

Remark 3 *If g(x) = x, then Wachnicki's theorem reduces to Flett's theorem.*