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Types of Infinity

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Abstract

The concept of infinity pervades much of modern mathematics. Even elementary school students use the word infinity. However, what do we mean by infinity? While, for many infinity merely means either very large or not finite, in various contexts, the term infinity carries more information. Here we consider infinity in the contexts of cardinality (sizes of sets), ordinals (the order of numbers), and real numbers (a number system). Our aim is to introduce these various notions of infinity to interested mathematics students and whet their appetite for more. To accomplish this, we have included three online applets for readers to consider when teaching or learning about mappings from between natural and rational numbers.

1 Introduction[1]

Teachers and students of mathematics can appreciate Buzz Lightyear's mathematical prowess and insight as he correctly states, "To infinity and beyond." Indeed, as we will see, Buzz gets it!

Even in elementary school, students use the word *infinity* to denote the notion of very large. Later in mathematics, students discuss infinite sets as well as infinities related to number systems. But, do we all mean the same thing by *infinity*? As we will see, infinity has a number of distinct yet interconnected meanings. For instance, infinity can be explored in the contexts of cardinals (sizes of sets), ordinals (orderings of sets), and real numbers (a number system). In this paper, we we present these ideas in a manner commensurate with the interests of inquisitive high school and undergraduate students and teachers. We hope that these ideas regarding infinity stimulate the reader to new interests in these intriguing concepts.

[1] In order to simplify several discussions, we freely assume the Axiom of Choice throughout this paper. This axiom is detailed later in this paper.

This paper is a soft investigation of concepts regarding infinity. For more detailed exposition, consider the following: [4] and [7]. In our companion paper [3] we consider the concept of ∞ and employ one- and two-point compactifications to investigate how the graphs of rational functions can pass through or bounce off ∞ in a manner similar to the graph of a polynomial passing through or bouncing off the x-axis.

2 Cardinal Numbers

Most simply stated, the cardinality of a set is the number of elements in that set. Let us begin with a definition which will be followed up with a couple of examples.

Definition 1. *Let A and B be sets. We say that A and B have the same* cardinality*, denoted* $\mathrm{card}(A) = \mathrm{card}(B)$*, if and only if there is a one-to-one and onto function (i.e., invertible function) whose domain is A and range is B.*

A function $f : A \to B$ is one-to-one if each element of A maps to a unique element of B, though there may be some extra elements of B which are not partnered with elements of A. A function $f : A \to B$ is onto if each element of B is mapped to by an element of A, though there may be some elements of B which are partnered with several elements of A. A function is invertible if it is both one-to-one and onto. Such a function makes sure every element of A is uniquely partnered with some element of B and vice-versa.

This means that sets of objects have the same size (i.e., cardinality) if we can match the elements between the sets in a one-to-one fashion. For example, $P = \{a, b, c\}$ and $Q = \{☺, ☹, \otimes\}$ both have the same cardinality: $\mathrm{card}(P) = \mathrm{card}(Q)$ because we can exhibit an invertible function between these sets: $a \leftrightarrow ☺$, $b \leftrightarrow ☹$, $c \leftrightarrow \otimes$. More simply stated: every element in set P maps to a distinct element in set Q and every element in Q maps to a distinct element in P (resulting in no unused elements in either direction). In this case, we might write $\mathrm{card}(P) = \mathrm{card}(Q) = 3$. Before moving on, let us be careful to point out that so far $\mathrm{card}(A)$ does not mean anything by itself. We define what it means for cardinalties to be the same and below we will define how one can compare cardinalities. But $\mathrm{card}(A)$ itself is not any kind of concrete object, yet.

When cardinalities are not the same, we can still compare them with inequalities. We write that $\mathrm{card}(A) \leq \mathrm{card}(B)$ if there is a function $f : A \to B$ which is one-to-one but possibly not onto. If there is a one-to-one function $f : A \to B$ but there is no invertible function between A and B, we write $\mathrm{card}(A) < \mathrm{card}(B)$. We might want to restate this as: If set B has at least one more element than set A, then $\mathrm{card}(A) < \mathrm{card}(B)$. However, while this is true for finite sets, it fails for infinite sets. When a set is infinite, adding in one more element does not change its cardinality!

It is easy to show[2] that if one has an onto function $f : A \to B$, then there must be a one-to-one function $g : B \to A$. This means that one-to-one and onto are dual to each other. An intuitive, yet a bit tricky to prove, theorem says that if $\mathrm{card}(A) \leq \mathrm{card}(B)$ and $\mathrm{card}(B) \leq \mathrm{card}(A)$, then $\mathrm{card}(A) = \mathrm{card}(B)$ (i.e., if neither set is bigger than the other, they must have the same size). This

[2]For each $x \in \mathrm{range}(f) = B$ choose some $g_x \in A$ such that $f(g_x) = x$ (can be done since f is onto). Then $g : B \to A$ given by $g(x) = g_x$ is a one-to-one function (if $g_x = g(x) = g(y) = g_y$ then $x = f(g_x) = f(g_y) = y$). Note that in proving this result, we used the Axiom of Choice.