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Numerical Solution of a One-Dimensional Nonlocal Helmholtz Equation by Perfectly Matched Layers

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Abstract. We consider the computation of a nonlocal Helmholtz equation by using perfectly matched layer (PML). We first derive the nonlocal PML equation by extending PML modifications from the local operator to the nonlocal operator of integral form. After that, we give stability estimates of some weighted-average values of the nonlocal Helmholtz solution and prove that (i) the weighted-average value of the nonlocal PML solution decays exponentially in PML layers in one case; (ii) in the other case, the weighted-average value of the nonlocal Helmholtz solution itself decays exponentially outside some domain. Particularly for a typical kernel function $\gamma_1(s) = \frac{1}{2}e^{-|s|}$, we obtain the Green's function of the nonlocal Helmholtz equation, and use the Green's function to further prove that (i) the nonlocal PML solution decays exponentially in PML layers in one case; (ii) in the other case, the nonlocal Helmholtz solution itself decays exponentially outside some domain. Based on our theoretical analysis, the truncated nonlocal problems are discussed and an asymptotic compatibility scheme is also introduced to solve the resulting truncated problems. Finally, numerical examples are provided to verify the effectiveness and validation of our nonlocal PML strategy and theoretical findings.

AMS subject classifications: 49M25, 65N22, 65N06, 65R20, 82C21, 46N40, 45A05

Key words: Nonlocal wave propagation, Helmholtz equation, perfectly matched layer, asymptotic compatibility scheme, Green's function.

1. Introduction

The development of nonlocal models has grown impressively over the last decade because of its huge potential of emerging applications in various research areas, such as the peridynamical theory of continuum mechanics, the nonlocal wave propagation,

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and the modeling of nonlocal diffusion process [7, 17, 33, 40, 45]. In this paper, we consider the computation of a nonlocal Helmholtz equation on the whole real axis

$$\mathcal{L}_\delta u(x) - k^2 u(x) = f(x), \quad x \in \mathbb{R}, \tag{1.1}$$

where k is a constant related to the traditional wavenumber for the local Helmholtz equation, the source $f \in L^2(\mathbb{R})$ is supported on $\Omega := (-l, l)$, and the nonlocal operator $\mathcal{L}_\delta$ is defined as

$$\mathcal{L}_\delta u(x) = \int_{\mathbb{R}} \big(u(x) - u(y)\big)\gamma_\delta(y - x)\mathrm{d}y. \tag{1.2}$$

The kernel function γ_δ in (1.2) is determined by a rescaling of a parent kernel γ_1 through

$$\gamma_\delta(s) = \frac{1}{\delta^3}\gamma_1\left(\frac{s}{\delta}\right), \tag{1.3}$$

where the parameter δ represents the range/radius of nonlocal interaction, and $\gamma_1(s) \in L^1(\mathbb{R})$ is piecewisely smooth, and satisfies (Ref. [16, 19, 35, 43])

- nonnegativeness: $\gamma_1(s) \geq 0$;
- symmetry in s: $\gamma_1(s) = \gamma_1(-s)$;
- finite horizon: $\exists\, l_\gamma > 0$, such that $\gamma_1(s) = 0$ if $|s| > l_\gamma$;
- the second moment condition $\frac{1}{2}\int_{\mathbb{R}} s^2\gamma_1(s)ds = 1$.

Recently, much works are carried out for the simulation of nonlocal problems with free or fixed boundary conditions. There are applications in which the simulation of an infinite medium may be useful, such as wave or crack propagation in whole space. The nonlocal Helmholtz equation can be used to describe the nonlocal wave propagation. In fact, it can be derived from the nonlocal wave equation

$$\left(\partial_t^2 + \mathcal{L}_\delta\right) u(x,t) = f(x,t), \quad x \in \mathbb{R}, \tag{1.4}$$

where $u(x,t)$ represents the displacement field, $f(x,t)$ is the space-time source term with compact support at all time. If we make the ansatz that $f(x,t)$ is a superposition of the time-harmonic sources $f(x)e^{-\mathrm{i}kt}$. Then, for each k, the corresponding mode $u(x)e^{-\mathrm{i}kt}$ satisfies

$$\mathcal{L}_\delta\big(u(x)e^{-\mathrm{i}kt}\big) - k^2 u(x)e^{-\mathrm{i}kt} = f(x)e^{-\mathrm{i}kt}.$$

Thus the time domain solution $u(x,t)$ is the sum of the time-harmonic modes $u(x)e^{-\mathrm{i}kt}$ over all possible values of k.

The aim of this paper is to develop an efficient computation of a nonlocal Helmholtz equation on the whole real axis. Absorbing boundary conditions (ABCs) are a successful approach to simulate the wave behaviors of a physical domain of interest by imposing a suitable boundary condition, to absorb the impinging wave at artificial boundaries. For the construction of tractional ABCs, it is well studied for local problems [1, 3, 22–