Numer. Math. Theor. Meth. Appl. Vol. **15**, No. 2, pp. 336-359
doi: 10.4208/nmtma.OA-2021-0154 May 2022

The Immersed Interface Hybridized Difference Method for Parabolic Interface Problems

Youngmok Jeon[1] and Son-Young Yi[2,*]

[1] *Department of Mathematics, Ajou University, Suwon 16499, Republic of Korea*

[2] *Department of Mathematical Sciences, The University of Texas at El Paso, El Paso, TX 79968, USA*

Received 9 September 2021; Accepted (in revised version) 26 February 2022

Abstract. We propose several immersed interface hybridized difference methods (IHDMs), combined with the Crank-Nicolson time-stepping scheme, for parabolic interface problems. The IHDM is the same as the hybrid difference method away from the interface cells, but the finite difference operators on the interface cells are modified to maintain the same accuracy throughout the entire domain. For the modification process, we consider virtual extensions of two sub-solutions in the interface cells in such a way that they satisfy certain jump equations between them. We propose several different sets of jump equations and their resulting discrete methods for one- and two-dimensional problems. Some numerical results are presented to demonstrate the accuracy and robustness of the proposed methods.

AMS subject classifications: 65N30, 65N38, 65N50

Key words: Parabolic interface problem, hybrid difference method, immersed interface.

1. Introduction

Interface problems are mathematical models using ordinary or partial differential equations that have discontinuous coefficients or singular source functions. They arise in a variety of disciplines, including mathematical biology, material sciences, fluid mechanics, and medicine. Some interesting applications include heart models [24,26], locomotion of aquatic animals [6,7], blood cell motion [31], biofilm formation [5,14,28], crystal growth [15], glacier prediction [19], electromigration of voids [20], and many more.

In general, the solutions to interface problems are not smooth or not even continuous at the interface. Therefore, desired numerical methods should be able to cap-

*Corresponding author. *Email addresses:* `yjeon@ajou.ac.kr` (Y. Jeon), `syi@utep.edu` (S.-Y. Yi)

ture such features of analytical solutions. Standard finite element, finite difference, and finite volume methods tend to lose accuracy near the interface unless they use interface-fitted meshes to resolve the low-regularity solution around the interface. In practice, however, structured meshes such as Cartesian grids that are independent of the interface are preferable. The advantage of using non-interface-fitted meshes is more prominent when the interface is geometrically complicated, or the dynamic simulation involves a moving interface, which requires repeated mesh generations.

In response to this need, various numerical methods have been developed based on non-interface-fitted meshes in the past decades, including immersed boundary (IB) method [24,25], immersed interface method (IIM) [8,16–18], extended finite element methods (XFEMs) [1,2], correction function method (CFM) [21,22], and kernel-free boundary integral method [29,30]. In addition to these methods, another method called the Immersed Interface Hybridized Difference Method (IHDM) was introduced by the author Jeon to solve elliptic interface problems on non-interface-fitted meshes [11]. The IHDM is closely related to the IIM and CFM, whose central idea is to modify the finite difference operator in a small region enclosing the interface to achieve higher-order global accuracy. In the IIM, the modified finite difference coefficients and correction terms are determined by the method of undetermined coefficients with a criterion to minimize the local truncation error. On the other hand, the CFM is based on the so-called correction function, which is defined as a solution of a partial differential equation (PDE) in the vicinity of the interface, paired with the given jump conditions at the interface. Once this PDE is solved by a minimization technique, the correction function is used to complete the finite difference discretization. As far as the IHDM is concerned, it modifies the finite difference operator near the interface after solving a PDE problem just like the CFM does, but the IHDM is different from the IIM and CFM in several aspects, as will be elaborated below.

The IHDM is based on the Hybrid Difference Method (HDM), a generalized finite difference method, which was first introduced by Jeon to study Poisson and Stokes equations [9] and further studied by him and his coauthors for other problems [10,12,13]. In the HDM framework, the numerical discretization procedure resembles that of the Hybridized Discontinuous Galerkin (HDG) method [3,4,23,27], in which the local problems from discretizing the governing equation in local cells are coupled through transmission conditions at the intercell boundary nodes. Therefore, the HDM is locally conservative, is amenable to static condensation, and can lend itself well to efficient solvers. Moreover, the extension of a low-order HDM to any arbitrarily high-order method is straightforward.

The IHDM is the same as the standard HDM away from the interface, but the finite difference operators are corrected on the cells containing the interface, which are called the interface cells. The derivation of the finite difference operators on the interface cells is based on the assumption that the solution of the interface problem is composed of smooth sub-solutions defined in the subdomains divided by the interface, and each sub-solution can be extended to an adjacent subdomain within a small neighborhood of the interface. In particular, these extended sub-solutions are supposed to