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An Efficient Mapped WENO Scheme Using Approximate Constant Mapping

Ruo Li[1,2] and Wei Zhong[2,3,*]

[1] *CAPT, LMAM, Peking University, Beijing 100871, China*
[2] *School of Mathematical Sciences, Peking University, Beijing 100871, China*
[3] *Northwest Institute of Nuclear Technology, Xi'an 710024, China*

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Abstract. We present a novel mapping approach for WENO schemes through the use of an approximate constant mapping function which is constructed by employing an approximation of the classic signum function. The new approximate constant mapping function is designed to meet the overall criteria for a proper mapping function required in the design of the WENO-PM6 scheme. The WENO-PM6 scheme was proposed to overcome the potential loss of accuracy of the WENO-M scheme which was developed to recover the optimal convergence order of the WENO-JS scheme at critical points. Our new mapped WENO scheme, denoted as WENO-ACM, maintains almost all advantages of the WENO-PM6 scheme, including low dissipation and high resolution, while decreases the number of mathematical operations remarkably in every mapping process leading to a significant improvement of efficiency. The convergence rates of the WENO-ACM scheme have been shown through one-dimensional linear advection equation with various initial conditions. Numerical results of one-dimensional Euler equations for the Riemann problems, the Mach 3 shock-density wave interaction and the Woodward-Colella interacting blastwaves are improved in comparison with the results obtained by the WENO-JS, WENO-M and WENO-PM6 schemes. Numerical experiments with two-dimensional problems as the 2D Riemann problem, the shock-vortex interaction, the 2D explosion problem, the double Mach reflection and the forward-facing step problem modeled via the two dimensional Euler equations have been conducted to demonstrate the high resolution and the effectiveness of the WENO-ACM scheme. The WENO-ACM scheme provides significantly better resolution than the WENO-M scheme and slightly better resolution than the WENO-PM6 scheme, and compared to the WENO-M and WENO-PM6 schemes, the extra computational cost is reduced by more than 83% and 93%, respectively.

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*Corresponding author. *Email addresses:* zhongwei2016@pku.edu.cn (W. Zhong), rli@math.pku.edu.cn (R. Li)

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1. Introduction

A number of essentially non-oscillatory (ENO) schemes [16–19, 33, 34] and weighted ENO (WENO) schemes [23, 27] have been developed quite successfully to solve the hyperbolic conservation laws, which may develop discontinuities in its solutions even if the initial conditions are smooth. The goal of this paper is to propose an improved version of the fifth-order WENO scheme for the hyperbolic conservation laws taking the form

$$\mathbf{u}_t + \sum_{\alpha=1}^{d} \mathbf{f}_\alpha(\mathbf{u})_{x_\alpha} = 0, \quad x_\alpha \in \mathbb{R}, \quad t > 0.$$

Here the function $\mathbf{u} = (u_1, u_2, \cdots, u_m)^{\mathrm{T}}$ is an m-dimensional vector of conserved variables, and flux $\mathbf{f}_\alpha(\mathbf{u})$ is a vector-valued function of m components with x_α and t variables.

Liu *et al.* [27] developed the first version of WENO schemes which convert an r-th order ENO scheme [16–19, 33, 34] into an $(r+1)$-th order WENO scheme by using a convex combination of all candidate substencils instead of just one as in the original ENO scheme. Later, Jiang and Shu [23] proposed the classic WENO-JS schemes with an improvement that an r-th order ENO scheme can be converted into a $(2r-1)$-th order WENO scheme by introducing a new definition of the smoothness indicator used to measure the smoothness of the numerical solution on a substencil. Then, the weighting method presented in [27] and the smoothness indicators designed in [23] eventually became a standard, and the WENO-JS schemes especially the fifth-order one [23] developed into one of the most popular high-order methods [25]. In recent decades, many successful works have been done to raise some issues about WENO schemes [1, 5, 6, 9, 11, 12, 20, 22, 44].

It was clearly pointed out by Henrick *et al.* [20] that, in general, the fifth-order WENO-JS scheme is only third-order or even less accurate at critical points of order $n_{\mathrm{cp}} = 1$ in smooth regions, where n_{cp} denotes the order of the critical point; e.g., $n_{\mathrm{cp}} = 1$ corresponds to $f' = 0, f'' \neq 0$ and $n_{\mathrm{cp}} = 2$ corresponds to $f' = 0, f'' = 0, f''' \neq 0$, etc. To overcome this problem, Henrick *et al.* [20] introduced a carefully designed mapping function leading to the first mapped WENO scheme named WENO-M. Compared to the WENO-JS scheme [23], the WENO-M scheme is able to recover the optimal convergence order near critical points in smooth regions and generate more accurate solutions. Another significant contribution of the work by Henrick *et al.* [20] is that they derived a strong sufficient condition on the weights of substencils for WENO schemes to achieve optimal convergence orders and this condition has become the primary criterion in the design of all other mapped WENO schemes [11, 12, 25, 26, 37, 38, 40]. Recently, Feng *et al.* [11] found that the mapping