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The Global Landscape of Phase Retrieval II: Quotient Intensity Models

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Abstract. A fundamental problem in phase retrieval is to reconstruct an unknown signal from a set of magnitude-only measurements. In this work we introduce three novel quotient intensity models (QIMs) based on a deep modification of the traditional intensity-based models. A remarkable feature of the new loss functions is that the corresponding geometric landscape is benign under the optimal sampling complexity. When the measurements $a_i \in \mathbb{R}^n$ are Gaussian random vectors and the number of measurements $m \geq Cn$, the QIMs admit no spurious local minimizers with high probability, i.e., the target solution x is the unique local minimizer (up to a global phase) and the loss function has a negative directional curvature around each saddle point. Such benign geometric landscape allows the gradient descent methods to find the global solution x (up to a global phase) without spectral initialization.

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1 Introduction

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1.1 Background

The intensity-based model for phase retrieval is

$$y_j = |a_j \cdot u|^2, \quad j = 1, \cdots, m,$$

where $a_j \in \mathbb{R}^n$, $j = 1, \cdots, m$ are given vectors and m is the number of measurements. The phase retrieval problem aims to recover the unknown signal $x \in \mathbb{R}^n$ based on the measurements $\{(a_j, y_j)\}_{j=1}^m$. A natural approach to solve this problem is to consider the minimization problem

$$\min_{u \in \mathbb{R}^n} \quad f(u) = \frac{1}{m} \sum_{j=1}^m ((a_j \cdot u)^2 - (a_j \cdot x)^2)^2. \tag{1.1}$$

However, as shown in [28], to guarantee the above loss function to have benign geometric landscape, the requirement of sampling complexity is $\mathcal{O}(n \log^3 n)$. This result is recently improved to $\mathcal{O}(n \log n)$ in [6]. On the other hand, due to the heavy tail of the quartic random variables in (1.1), such results seem to be optimal for this class of loss functions.

To remedy this issue, we propose in this work three novel quotient intensity models (QIMs) to recover x under optimal sampling complexity. We rigorously prove that, for Gaussian random measurements, those empirical loss functions admit the benign geometric landscapes with high probability under the optimal sampling complexity $\mathcal{O}(n)$. Here, the phrase "benign" means: (1) the loss function has no spurious local minimizers; and (2) the loss function has a negative directional curvature around each saddle point. The three quotient intensity models are

QIM1:

$$\min_{u \in \mathbb{R}^n} f(u) = \frac{1}{m} \sum_{k=1}^m \frac{((a_k \cdot u)^2 - (a_k \cdot x)^2)^2}{(a_k \cdot x)^2}. \tag{1.2}$$

QIM2:

$$\min_{u \in \mathbb{R}^n} f(u) = \frac{1}{m} \sum_{k=1}^m \frac{((a_k \cdot u)^2 - (a_k \cdot x)^2)^2}{\beta \|u\|_2^2 + (a_k \cdot x)^2}. \tag{1.3}$$

QIM3:

$$\min_{u \in \mathbb{R}^n} f(u) = \frac{1}{m} \sum_{k=1}^m \frac{((a_k \cdot u)^2 - (a_k \cdot x)^2)^2}{\|u\|_2^2 + \beta_1 (a_k \cdot u)^2 + \beta_2 (a_k \cdot x)^2}. \tag{1.4}$$