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Morse Index of Multiple Blow-Up Solutions to the Two-Dimensional Sinh-Poisson Equation

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Abstract. In this paper we consider the Dirichlet problem

$$\begin{cases} -\Delta u = \rho^2(e^u - e^{-u}) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where ρ is a small parameter and Ω is a C^2 bounded domain in $\mathbb{R}^2$. In [1], the author proves the existence of a m-point blow-up solution u_ρ jointly with its asymptotic behaviour. We compute the Morse index of u_ρ in terms of the Morse index of the associated Hamilton function of this problem. In addition, we give an asymptotic estimate for the first $4m$ eigenvalues and eigenfunctions.

Key Words: Morse index, sinh-Poisson equation, eigenvalues estimates.

AMS Subject Classifications: 35P30, 35B40

1 Introduction

We are concerned with the study of the Morse index of the Dirichlet problem

$$\begin{cases} -\Delta u = \rho^2(e^u - e^{-u}) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where ρ is a small parameter and Ω is a C^2 bounded domain in $\mathbb{R}^2$. This equation has been widely studied as it is strictly related to the vortex-type configuration for 2D turbulent Euler flows (see [6, 7, 24]). Its importance is due to the fact that, suitably adapted, it describes interesting phenomena in widely different areas like liquid helium, meteorology and oceanography; it highlights effects that are important in all those subjects. Moreover, its dynamics are isomorphic to those of the electrostatic guiding-center plasma,

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which have been widely extended to describe strongly magnetized plasmas (see for instance [28,31,32]).

It has been known since Kirchhoff [21] that, if we let $\xi_i \in \Omega$, $i = 1, \cdots, m$, be the centres of the vorticity blobs, then the ξ_i's obey an approximate Hamiltonian dynamic associated to the Hamiltonian function

$$\mathcal{F}(\xi_1, \cdots, \xi_m) = \frac{1}{2}\sum_{k=1}^{m} R(\xi_k) + \frac{1}{2}\sum_{1\le k,j\le m} \alpha_k \alpha_j G(\xi_k, \xi_j), \tag{1.2}$$

with $\alpha_k \in \{1, -1\}$, $k = 1, \cdots, m$, depending on the sign of the corresponding vorticity blob. Through two different approaches, Joyce [19] and Montgomery [27] proved at heuristic level that, if we let ω be the vorticity, ψ the flow's steam function, $\beta \in \mathbb{R}$ the inverse of the temperature, and $Z > 0$ an appropriate normalization constant, then we have

$$\omega(\psi) = \frac{2}{Z}\sinh(-\psi)$$

for a flow with total vorticity equal to zero, i.e.,

$$\int_\Omega \omega = 0 \quad \text{and} \quad \omega(\psi) = \frac{1}{Z}e^{-\beta\psi}$$

for a flow with total vorticity equal to one, i.e., $\int_\Omega \omega = 1$. Setting $u = -|\beta|\psi$, the 2D Euler equation in stationary form

$$\begin{cases} \mathbf{w}\cdot\nabla\omega(\psi) = 0 & \text{in } \ \Omega, \\ -\Delta\psi = 0 & \text{in } \ \Omega, \\ \mathbf{w}\cdot\nu = 0 & \text{on } \ \partial\Omega, \end{cases} \tag{1.3}$$

where $\mathbf{w}$ is the velocity field, reduces to the sinh-Poisson equation (1.1), where $\rho = \sqrt{\frac{|\beta|}{Z}}$. More recently, in [4,5] was rigorously proved that for any $\beta \ge -9\pi$, $\omega(\psi) = \frac{1}{Z}e^{-\beta\psi}$ is the mean field-limit vorticity for both the micro-canonical and the canonical equilibrium statistic distributions for the Hamiltonian point-vortex model. The solutions to (1.3) with $\beta < 0$ of the type of those suggested in [19] ('negative temperature' states) have shown to represent very well the numerical experiment on the Navier-Stokes equations with high Reynolds number [25,26,29]. For further details and recent developments in the study of this problem see [5,22,34]. Due to the just mentioned results much effort has been put into finding out explicit solutions for the Euler equations with Joy-Montgomery vorticity. Among the most relevants there are the Mallier-Maslowe [23] counter rotating vortices, and their generalization [8,9]. The Mallier-Maslowe vortices are sign changing solutions to $-\Delta u = \rho^2\sinh(u)$, with 1-periodic boundary conditions, one absolute maxima and minima and two nodal domains in each periodic cell, the resulting Euler flow is composed of symmetric and disjoint regions where the velocity fields are counter directed.