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A Holomorphic Operator Function Approach for the Laplace Eigenvalue Problem Using Discontinuous Galerkin Method

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Abstract. The paper presents a holomorphic operator function approach for the Laplace eigenvalue problem using the discontinuous Galerkin method. We rewrite the problem as the eigenvalue problem of a holomorphic Fredholm operator function of index zero. The convergence for the discontinuous Galerkin method is proved by using the abstract approximation theory for holomorphic operator functions. We employ the spectral indicator method to compute the eigenvalues. Extensive numerical examples are presented to validate the theory.

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1 Introduction

The Laplace eigenvalue problem has many applications such as vibration modes in acoustics, nuclear magnetic resonance measurements of diffusive transport, electron wave functions in quantum waveguides, construction of heat kernels in the theory of diffusion [14]. The results for the Laplace eigenvalue problem are useful in the analysis of many other eigenvalue problems. The problem has been studied by many researchers, see e.g., [8]. The theory and numerical methods are well-developed. There are many different finite element methods for the Laplace eigenvalue problem in literature [1,6,7,20].

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For the conforming finite element method using the standard Lagrange elements, the convergence analysis follows directly the theory of Babuška and Osborn [4].

The discontinuous Galerkin (DG) method has been first introduced in the seventies of the last century for the numerical approximation of hyperbolic problems [22], and independently, in the context of elliptic and parabolic equations [2, 11]. Relaxing the continuity of approximation functions across the finite element boundaries allows the DG methods to be easily implemented on highly unstructured meshes. The locality and flexibility also make the methods well suited for parallelization and applications of domain decomposition techniques. DG methods can also be used for spectral computations of the Laplace equation [1]. Due to the lack of the norm convergence of the solution operator, DG methods have not fit into the abstract convergence theory of Babuška and Osborn [4]. The spectral correctness is analyzed in [1] along the lines of [10], using the fact that DG is a non-conforming discretization of an operator with compact inverse.

In this paper, we take a different method to prove the convergence of the Laplace eigenvalue problem using DG methods. We reformulate the Laplace eigenvalue problem as an eigenvalue problem of a holomorphic operator function. Then DG method and the spectral projection are used to compute the eigenvalues inside a region on the complex plane. Using the fundamental properties of DG [21] for the source problem and the approximation results for the eigenvalues of holomorphic Fredholm operator functions [5, 18, 19], we prove the convergence of DG approximation. The method can be easily extended to the biharmonic eigenvalue problem and it can be parallelized to compute many eigenvalues effectively.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries for the problem. Section 3 gives the discontinuous Galerkin method and some fundamental properties. Section 4 gives the error estimate of the eigenvalue problem. The spectral indicator method is introduced in Section 5. Numerical examples are presented in the last section.

2 Preliminaries

In this section, some preliminaries on the eigenvalue approximation theory of holomorphic Fredholm operator functions [18, 19] are presented.

We first give some notations. Let X,Y be complex Banach spaces and $\Omega\subset\mathbb{C}$ be a compact simply connected region. Then we denote $\mathcal{L}(X,Y)$ the space of bounded linear operators from X to Y. Let $F:\Omega\to\mathcal{L}(X,Y)$ be a holomorphic operator function on Ω.

Definition 2.1. *(cf.* [12]*) A bounded linear operator $F:X\to Y$ is said to be Fredholm if*

1. the subspace $\mathcal{R}(F)$ (range of F) is closed in Y;

2. the subspace $\mathcal{N}(F)$ (null space of F) and $Y/\mathcal{R}(F)$ are finite-dimensional.

The index of F is the integer defined by

$$ind(F)=dim\mathcal{N}(F)-dim(Y/\mathcal{R}(F)). \tag{2.1}$$