Analysis in Theory and Applications
Anal. Theory Appl., Vol. **37**, No. 3 (2021), pp. 426-464

DOI: 10.4208/ata.2021.lu80.09
September 2021

Intrinsic Square Function Characterizations of Several Hardy–Type Spaces–a Survey

Xianjie Yan, Dachun Yang and Wen Yuan*

Laboratory of Mathematics and Complex Systems (Ministry of Education of China), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China

Received 25 October 2020; Accepted (in revised version) 21 July 2021

Dedicated to Prof. Shanzhen Lu with admiration on the occasion of his 80th birthday

---

Abstract. In this article, the authors give a survey about the recent developments of intrinsic square function characterizations and their applications on several Hardy-type spaces, including (weak) Musielak–Orlicz Hardy spaces, variable (weak) Hardy spaces, and Hardy spaces associated with ball quasi-Banach function spaces. The authors also present some open problems.

Key Words: Intrinsic square function, (weak) Musielak–Orlicz Hardy space, variable (weak) Hardy space, ball quasi-Banach function space, Campanato space.

AMS Subject Classifications: 42B25, 42B30, 42B35, 46E30

---

1 Introduction

In order to settle a conjecture proposed by Fefferman and Stein [8] on the boundedness of the Lusin area function $S(f)$ from the weighted Lebesgue space $L^2_{\mathcal{M}(v)}(\mathbb{R}^n)$ to the weighted Lebesgue space $L^2_v(\mathbb{R}^n)$ with $0 \le v \in L^1_{\rm loc}(\mathbb{R}^n)$, where $\mathcal{M}(v)$ denotes the Hardy–Littlewood maximal function of v, Wilson originally introduced the intrinsic square functions in [43] and obtained their boundedness on the weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ in [44], where $p \in (1,\infty)$ and w belongs to the Muckenhoupt weight $A_p(\mathbb{R}^n)$. Later, Huang and Liu [15] established the intrinsic square function characterizations of the weighted Hardy space $H^1_w(\mathbb{R}^n)$ with $\alpha \in (0,1]$ and $w \in A_{1+\alpha/n}(\mathbb{R}^n)$, under the additional assumption that $f \in L^1_w(\mathbb{R}^n)$. This was further generalized to the weighted Hardy space $H^p_w(\mathbb{R}^n)$ with $\alpha \in (0,1]$, $p \in (n/(n+\alpha),1)$, and $w \in A_{p(1+\alpha/n)}(\mathbb{R}^n)$ by Wang and Liu [41], under the additional assumption that $f \in (\mathrm{Lip}(\alpha,1,0))^*$, where $(\mathrm{Lip}(\alpha,1,0))^*$ denotes the dual space of the Lipschitz space $\mathrm{Lip}(\alpha,1,0)$. In addition, Wang and Liu

*Corresponding author. *Email addresses:* `xianjieyan@mail.bnu.edu.cn` (X. Yan), `dcyang@bnu.edu.cn` (D. Yang), `wenyuan@bnu.edu.cn` (W. Yuan)

in [40] proved some weak type estimates of intrinsic square functions on the weighted Hardy space $H_w^p(\mathbb{R}^n)$ with $\alpha \in (0,1)$, $p = n/(n+\alpha)$, and $w \in A_1(\mathbb{R}^n)$; Wang [38] obtained the boundedness of the intrinsic square functions including the g_λ^*-function on the weighted weak Hardy space $WH_w^p(\mathbb{R}^n)$ with $\lambda \in (3+2\alpha/n,\infty)$, where $\alpha \in (0,1]$, $p \in (n/(n+\alpha),1]$, and $w \in A_{p(1+\alpha/n)}(\mathbb{R}^n)$. Indeed, these intrinsic square functions can be thought of as "grand maximal" square functions in the style of the "grand maximal function" of Fefferman and Stein from [8]; they dominate all the square functions of the form $S(f)$ (and the classical ones as well), but are not essentially bigger than any one of them. Especially, the intrinsic Lusin area function has the distinct advantage of being pointwisely comparable at different cone openings, which is a property long known not to be true for the classical Lusin area function; see Wilson [43–46] and also Lerner [20,21].

With the development of the real-variable theories of several Hardy-type spaces on Euclidean spaces, the study of the intrinsic square functions on these spaces has attracted a lot of attentions in recent years. Liang and Yang in [25] first introduced the s-order intrinsic square functions and characterized the Musielak–Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$ in terms of the related intrinsic Lusin area function, the intrinsic g-function, and the intrinsic g_λ^*-function with the best-known range $\lambda \in (2+2(\alpha+s)/n,\infty)$, which essentially improve the known results in [15] and [41]. Motivated by this, Zhuo et al. [57] generalized the corresponding results in [25] to the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ with $\lambda \in (3+2(\alpha+s)/n,\infty)$; Yan [47, 48] obtained similar characterizations on the weak Musielak–Orlicz Hardy space $WH^\varphi(\mathbb{R}^n)$ and the variable weak Hardy space $WH^{p(\cdot)}(\mathbb{R}^n)$. Very recently, Yan et al. [49] continued the above line of research and established the intrinsic square function characterizations of the Hardy type space $H_X(\mathbb{R}^n)$ related to a ball quasi-Banach function space X satisfying some mild additional assumptions. For more applications of such intrinsic square functions, we refer the reader to [10,11,23,37,39,52].

In this article, we first give a survey on the recent developments of intrinsic square function characterizations and their applications on several Hardy-type spaces, including (weak) Musielak–Orlicz Hardy spaces, variable (weak) Hardy spaces, and Hardy spaces associated with ball quasi-Banach function spaces. To be precise, the main results that we review include: the (finite) atomic characterizations of the Musielak–Orlicz Hardy space, the atomic characterization of the variable Hardy space, the (finite) atomic characterizations of the Hardy space associated with ball quasi-Banach function space, Campanato type spaces, and duality theories related to the above three kinds of function spaces as well as their intrinsic square function characterizations. We also correct some errors and seal some gaps existing in [49, Theorems 1.10, 1.12, 1.15, and 1.16]. Finally, we present some open problems.

To be precise, the remainder of this survey is organized as follows.

In Section 2, we recall the definitions of intrinsic square functions and some notation which are used throughout this article.

The aim of Section 3 is to review the intrinsic square function characterizations of $H^\varphi(\mathbb{R}^n)$ and $WH^\varphi(\mathbb{R}^n)$, where $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ satisfies that, for any given $x \in \mathbb{R}^n$, $\varphi(x,\cdot)$ is an Orlicz function and $\varphi(\cdot,t)$ is a Muckenhoupt $\mathbb{A}_\infty(\mathbb{R}^n)$ weight uniformly