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Homogenization of Stationary Navier-Stokes Equations in Domains with 3 Kinds of Typical Holes

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Abstract. The aim of this paper is to investigate homogenization of stationary Navier-Stokes equations with a Dirichlet boundary condition in domains with 3 kinds of typical holes. For space dimension $N=2$ and 3, we utilize a unified approach for 3 kinds of tiny holes to accomplish the homogenization of stationary Navier-Stokes equations. The unified approach due to Lu [1] is mainly based on the uniform estimates with respect to ε for the generalized cell problem inspired by Tartar.

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1 Introduction

It is well known homogenization describes the asymptotic behavior of fluid flows in perforated domains, as the number of holes increases and goes to infinity, the size of holes will go to zero, the flow will tend to the solution of certain effective or "homogenized" equations which are homogeneous in form.

Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with C^1 boundary, here and hereafter $N=2$ or 3. The domain Ω is covered with a regular mesh of size ε, each cell being a cube Q_k. At the center of each cube Q_k included in Ω there is a hole $T_{\varepsilon,k}$ such that

$$B(\varepsilon x_k,\delta_1 a_\varepsilon)\subset\subset T_{\varepsilon,k}=\varepsilon x_k+a_\varepsilon T\subset\subset B(\varepsilon x_k,\delta_2 a_\varepsilon)\subset\subset B(\varepsilon x_k,\delta_3 a_\varepsilon)\subset\subset Q_k,$$

where $B(x,\delta)$ stands for a small open ball of radius δ centered at x, $Q_k:=(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})^N+\varepsilon k$, and $x_k=x_0+k$ with $x_0\in(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})^N$, for each $k\in\mathbb{Z}^N$. Here the model hole T is assumed to be a closed, bounded, and simply connected set with C^1 boundary, rescaled at size a_ε. δ_i

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$(i=1,2,3)$ are positive constants. The mutual distance of holes is denoted by ε, the size of holes is denoted by a_ε, and $\varepsilon x_k = \varepsilon x_0 + \varepsilon k$ locates the holes. Without loss of generality, we assume that $x_0 = 0$ and $0 < a_\varepsilon < \varepsilon \leq 1$.

The open set Ω_ε is obtained by removing all the holes $T_{\varepsilon,k}$ from Ω, so Ω_ε is also a bounded C^1, Precisely, Ω_ε is defined as following:

$$\Omega_\varepsilon = \Omega \setminus \bigcup_{k \in K_\varepsilon} T_{\varepsilon,k}, \quad \text{where} \quad K_\varepsilon = \Big\{ k \in \mathbb{Z}^N : \bar{Q}_k \subset \Omega \Big\}. \tag{1.1}$$

Consider the Dirichlet problem of the stationary Navier-Stokes equations with external force $\mathbf{f} \in [L^2(\Omega)]^N$ in Ω_ε:

$$\begin{cases} \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}_\varepsilon - \nu \Delta \mathbf{u}_\varepsilon + \nabla p_\varepsilon = \mathbf{f}, & \text{in } \Omega_\varepsilon, \\ \operatorname{div} \mathbf{u}_\varepsilon = 0, & \text{in } \Omega_\varepsilon, \\ \mathbf{u}_\varepsilon = 0, & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{1.2}$$

where $\mathbf{u}_\varepsilon$ denotes the velocity of the flow, p_ε stands for the pressure, and $\nu > 0$ is the viscosity coefficient. For each fixed $\varepsilon > 0$, the existence of the weak solution $(\mathbf{u}_\varepsilon, p_\varepsilon) \in [H_0^1(\Omega_\varepsilon)]^N \times L_0^2(\Omega_\varepsilon)$ to (3.9) has been established in [2], where $H_0^1(\Omega_\varepsilon)$ stands for the usual Sobolev space with zero trace, and $L_0^2(\Omega_\varepsilon)$ is the collection of all L^2 functions with zero average.

In recent decades, due to its physical importance, complexity, rich phenomena, and mathematical challenges, there have been a lot of literatures on homogenization problem. Allaire [3, 4] did a systematic study on the Stokes equations with a Dirichlet boundary condition in a domain containing many tiny holes, which are periodically distributed in each direction of the axes. For holes of critical size, Allaire established an abstract framework and showed that the limit problem is described by a law of Brinkman type (see [5]). He also proved that for smaller holes, the limit problem reduces to the Stokes equations, and for larger holes, to Darcy's law. Similar as in [3], we define the ratio σ_ε between the size and the mutual distance of the holes:

$$\sigma_\varepsilon = \left(\frac{\varepsilon^3}{a_\varepsilon} \right)^{\frac{1}{2}}, \quad N=3; \quad \sigma_\varepsilon = \varepsilon \left| \log \frac{a_\varepsilon}{\varepsilon} \right|^{\frac{1}{2}}, \quad N=2. \tag{1.3}$$

On the other hand, different from Allaire's framework, Tartar [6] utilized the so-called cell problem to study the asymptotic behavior of the solution family $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ to Dirichlet problem of Stokes equations as $\varepsilon \to 0$, under the assumption that the size of the holes is proportional to the mutual distance of the holes, i.e.

$$a_\varepsilon = a_* \varepsilon \qquad \text{for some } a_* > 0. \tag{1.4}$$

Later, homogenization problems of fluid flows are generalized to more complex models. When the size of holes is proportional to the mutual distance of holes, for the incompressible Navier-Stokes equations, for the compressible Navier-Stokes equations, and for