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Coincidence Point Theorems, Intersection Theorems and Saddle Point Theorems on FC-spaces*

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Abstract: In this paper, we first give the definitions of finitely continuous topological space and FC-subspace generated by some set, and obtain coincidence point theorem, whole intersection theorems and Ky Fan type matching theorems, and finally discuss the existence of saddle point as an application of coincidence point theorem.

Key words: FC-space, FC-subspace generated by set, saddle point
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1 Preliminaries

Let X and Y be two topological spaces, a multimap $T:\ X \multimap Y$ be a map from X to the power set 2^Y of Y. And let

$$T(A) = \bigcup_{x\in A} T(x), \qquad A \subset X.$$

Define $T^-:\ Y \multimap X,\ T^-y = \{x \in X : y \in T(x)\}$ for each $y \in Y$.

Obviously, $y \in T(x) \Longleftrightarrow x \in T^-y$.

Give a class $\mathfrak{A}$ of maps, let $\mathfrak{A}(X, Y) = \{F :\ X \multimap Y \text{ and } F \in \mathfrak{A}\}$. We denote by $\mathfrak{A}_c$ the set of finite composites of maps in $\mathfrak{A}$.

A class $\mathfrak{A}$ of maps is defined by the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;

(ii) Each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued;

(iii) For any polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces are suitably chosen.

Note that a polytope is a convex hull of a nonempty finite subset of a vector space with the Euclidean topology.

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Furthermore, a class $\mathfrak{A}_c^k(X,Y)$ is defined as follows:

$F \in \mathfrak{A}_c^k(X,Y) \Longleftrightarrow$ for any compact subset K of X, there exists a $G \in \mathfrak{A}_c(K,Y)$ such that $G(x) \subset F(x)$ for each $x \in K$. This map is called to be admissible.

$F \in \mathbb{V}(X,Y) \Longleftrightarrow F$ is an acyclic map, that is, F is a u.s.c map with compact acyclic values.

We know that $\mathbb{V}(X,Y) \subset \mathfrak{A}(X,Y) \subset \mathfrak{A}_c^k(X,Y)$ (see [1]).

Definition 1.1[2] *$(X,\ D,\ \{\phi_N\})$ is said to be a finitely continuous topological space (simply, FC-space), if X is a topological space, D is a nonempty subset of X and for each $N = \{x_0, x_1, \cdots, x_n\} \in \langle D\rangle$, there exists a continuous map $\phi_N : \triangle_n \to X$, where $\langle D\rangle$ denotes the set of all nonempty finite subset of D. If $D = X$, then $(X,\ D,\ \{\phi_N\})$ is written as $(X,\ \{\phi_N\})$.*

Definition 1.2[3] *Let $(X,\ D,\ \{\phi_N\})$ be an FC-space, A and B be two subsets of X. B is said to be an FC-subspace of X with respect to A, if for each $N = \{x_0, x_1, \cdots, x_n\} \in \langle D\rangle$ and for each $\{x_{i_0}, x_{i_1}, \cdots, x_{i_k}\} \subset A \cap \{x_0, x_1, \cdots, x_n\}$, $\phi_N(\triangle_k) \subset B$, where $\triangle_k = \mathrm{co}(\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\})$. If $A = B$, then B is called an FC-subspace of X.*

Definition 1.3 *Let $(X,\ D,\ \{\phi_N\})$ be an FC-space, A be a nonempty subset of X. Define*

$$\mathrm{span}(A) := \bigcap\{Z \subset X :\ Z \text{ is an FC-subspace of } X \text{ containning } A\}.$$

span(A) has the following two properties.

Theorem 1.1 *Let $(X,\ D,\ \{\phi_N\})$ be an FC-space, A be a nonempty subset of X. Then span(A) is the smallest FC-subspace of X containing A.*

Proof. By Definition 1.3, we only need to prove that span(A) is an FC-subspace of X. In fact, for each $N = \{x_0, x_1, \cdots, x_n\} \in \langle D\rangle$ and any $M = \{x_{i_0}, x_{i_1}, \cdots, x_{i_k}\} \subset N \cap \mathrm{span}(A)$ and any FC-subspace Z of X containing A, since $M \subset N \cap Z$, then $\phi_N(\triangle_k) \subset Z$ by Definition 1.2. Therefore $\phi_N(\triangle_k) \subset \mathrm{span}(A)$. This means that span$(A)$ is an FC-subspace of X.

Remark 1.1 span(A) is said to be an FC-subspace of X generated by A.

Theorem 1.2 *Let $(X,\ D,\ \{\phi_N\})$ be an FC-space, A be a nonempty subset of X. Then* $\mathrm{span}(A) = \bigcup\{\mathrm{span}(B) :\ B \in \langle A\rangle\}$.

Proof. Let $L = \bigcup\{\mathrm{span}(B) : B \in \langle A\rangle\}$. Then for each $B \in \langle A\rangle$, $B \subset A \subset \mathrm{span}(A)$ and therefore $\mathrm{span}(B) \subset \mathrm{span}(A)$ by Theorem 1.1, so that $L \subset \mathrm{span}(A)$.

On the other hand, for each $x \in X$, $\{x\} \subset \mathrm{span}(\{x\})$, hence $A \subset L$.

Next we prove that L is an FC-subspace of X.

Indeed, for each $N = \{x_0, x_1, \cdots, x_n\} \in \langle D\rangle$ and any $\{x_{i_0}, x_{i_1}, \cdots, x_{i_k}\} \subset N \cap L$, since each $x_{i_j} (j = 0, 1, \cdots, k)$ satisfies $x_{i_j} \in L$, so there exists $B_{i_j} \in \langle A\rangle$ such that $x_{i_j} \in \mathrm{span}(B_{i_j})$. Let $\widehat{B} = \bigcup\limits_{j=0}^{k} B_{i_j}$. Then $\widehat{B} \in \langle A\rangle$ and $x_{i_j} \in \mathrm{span}(B_{i_j}) \subset \mathrm{span}(\widehat{B}) (j = 0, 1, \cdots, k)$.